Discovering Algebra
An Investigative Approach

A Guide for Parents
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Discovering Algebra: An Investigative Approach covers the topics offered in traditional algebra courses, but the teaching style as well as the learning experience might be different from what you remember from your own high school algebra course.

In the past, and probably in your own school experience, students were asked to spend a lot of time manipulating symbols—moving x’s, y’s, and numbers around in expressions and equations—before they got the chance to understand what they were doing. For example, you might recognize this scenario: After going over homework, your teacher showed a new type of problem and a method for solving it. You worked alone with pencil and paper and practiced solving problems of that type. For homework, you worked on more problems of the same type. The next day, the class went through that same process with a new type of problem. At some point, you took a test with a lot of problems on it. You had to remember the methods and figure out what method to use for each problem. If you did well on all the tests, you were “good at math.” If you didn’t do well, you might have thought you “just couldn’t do math.”

Many students cannot succeed in such an environment. Perhaps you had a hard time yourself. The teacher and textbook cannot furnish enough examples to apply to every new situation or problem. As a result, many students are limited in their understanding, unable to do more than mechanical manipulations. They don’t know when to apply a particular problem-solving strategy. They don’t come away from their math course with a set of ideas that weave together into “the big picture.” They doubt that mathematics will be relevant to their careers and they don’t see what others like about it. Even students who pass are reluctant to continue on in mathematics. Some develop “math phobia”—the fear of math—and avoid courses in science or business that require math. Ultimately, their fear limits their career choices and their life income.

But all students can learn math better, have a good time doing it, and come away with an appreciation of its value as a tool for science, business, and everyday life. Discovering Algebra is a program that helps all students reach a deep understanding of math by encouraging them to investigate interesting problems in cooperative groups, use technology where appropriate, and practice skills that make routine problems automatic.

All Students Benefit

From their own teaching experience, the Discovering Algebra authors know that all students can experience more success in mathematics. When the focus is on understanding concepts and problem-solving strategies instead of just memorizing formulas and procedures, students with concentration, attention, and memory issues can be more successful. Passive or reluctant students will learn to communicate better. To say that all students can learn math does not mean the course has been watered down. In fact, even very successful math students will find they are challenged, learn more, and remember longer with the Discovering Algebra approach. That’s because the concepts and methods are not isolated from
real-world applications, or from previously learned ideas, or from information they are getting in other classes. The mathematics that students study is closer to what is needed by both students seeking employment after high school and students preparing to attend college.

**Deep Understanding Is Important**

In your own math classes, you might have been told: “Just do it—don’t ask why.” But there are logical reasons behind mathematical methods and ideas, and the people who understand these reasons succeed at math and, ultimately, at science and business. *Discovering Algebra* helps more students understand these reasons. Because the concepts make sense to students, students remember the methods (or reinvent them if they’ve forgotten them) and can apply them to new problems. To help develop that kind of flexible understanding, *Discovering Algebra* offers a more visual approach, with clearer and more frequent illustrations and graphs, and thoughtful integration of text captions to lead students through examples. *Discovering Algebra* also acknowledges the need for a gradual development of mathematical ideas. Students are helped to see where the text is leading, and full-blown explanations are delayed until all the groundwork is laid. Once a topic has been made part of what students are expected to know, it is reviewed and referred to again whenever appropriate. Understanding the math can make the math more fun, will increase pride and confidence, bolster capacity for critical and abstract thinking, and increase the chance that students will use math in their lives.

**Students Learn Better in Cooperative Groups**

Students are not expected to do all this learning by themselves. Many students make sense of mathematical ideas best in interaction with other people, using informal language. They think best out loud, or they get ideas from others. And they understand better from seeing other students’ viewpoints. They learn that nothing bad happens if they make mistakes or misapply a procedure and that trial and error is also a respected strategy. This helps quiet or insecure students learn to contribute. When students are working in groups, the teacher circulates and observes, poses questions, and intervenes when necessary to assist. He or she works as a partner to student groups, monitoring the back-and-forth, modeling good communication, and drawing out clues that students are confused or on the right track. Group work helps students learn better, and it also teaches essential teamwork skills. In their groups, students will be asked to demonstrate their understanding both orally and in writing.

**Investigation Is Motivating**

Some students learn better by seeing, some by hearing, and some by reading, so an explanation that makes sense to one student might not make sense to another. These different “learning styles” are addressed by the investigations in *Discovering Algebra*. Because most students are more interested in class if the problems they investigate are related to the real world, many of the hands-on investigations involve problems that students might see in their lives outside school. Some investigations use very familiar scenarios, and others are career-oriented. Some investigations allow students to get up and move around, many use graphing calculators or other technology, and some involve pure mathematical ideas. In your student’s *Discovering Algebra* textbook is a career-oriented investigation on
page 103, an on-your-feet investigation on page 172 that uses motion sensors, and an activity using a familiar object—the bicycle—on page 132. The Multiply and Conquer Investigation on page 97 is an example of a pure mathematics activity. The teacher might have students work in groups on the investigation, and later lead a whole-class discussion. Each student develops his or her own understanding and benefits from sharing ideas and suggestions offered by others. Students learn that there are many approaches to solving problems. They also learn that they are individually responsible for describing orally or in writing what they have learned.

**Problem Solving Is Important**

In life, we all need to be good at solving problems that don’t exactly fit into a model we know. This is an important job skill and career asset as well: People who “think outside the box” to solve problems at work move up faster and are seen as leaders. To help prepare students to use math in their lives, many investigations in *Discovering Algebra* pose problems that students haven’t already been told how to solve. They learn to brainstorm, consider subproblems, come at a problem from a unique angle, and make diagrams and models. In this way, they learn problem-solving skills rather than learning how to solve only particular types of problems.

**Using Technology Helps**

Computers and calculators surround us, and students will use them at work, sometimes with custom-designed software, so working with them in these classes teaches students skills that will be useful later on. Whether you are computer literate yourself or strictly “low-tech,” your student is probably fascinated with technology, and using technology in class will help keep your student interested.

Technology is not used as a substitute for learning basic arithmetic. When used appropriately, technology can make mathematics more visual, more logical, and more fun. Most importantly, technology tools allow students to investigate many more situations and examples than they can explore by using pencil and paper. Getting fast results on numerous examples helps students see patterns, form generalizations, and test conclusions. That leads to a deeper understanding of concepts and a greater willingness to explore further and tackle larger problems.

If your student’s teacher does not have access to technology, or doesn’t have access on certain days, the *Discovering Algebra Teacher’s Edition* suggests low-tech alternatives for technology-dependent investigations. Calculator Notes for various calculator models are available online at www.keymath.com. These notes give the keystroke instructions to perform the functions needed for class activities.

Homework exercises that require a graphing calculator are noted in the text. The teacher can give you advice about which graphing calculator to purchase if you decide to buy one for your student.

**Practicing Skills Is Essential**

As students investigate a new concept, they develop and practice new skills. After students learn why a process works, they apply their new skills in the Practice Your Skills exercises in the student text. They extend these skills in the Reason and Apply exercises. Finally, each lesson has Review exercises so that students retain and extend their understanding of skills and concepts they learned in previous lessons. For further practice, your student’s teacher has probably received
a copy of Discovering Algebra: More Practice Your Skills. You can access these worksheets online at www.keymath.com/DA.

Discovering Algebra supports an approach to mathematics that brings about better understanding of concepts and skills. Instead of solving one type of problem after another, students engage in investigations, examples, and exercises that help them build up their own bank of skills and concepts. Students learn to describe how and why something is true. Instead of working alone, students bounce ideas off their peers. Because students are actively involved in acquiring skills and concepts, they can successfully attack and deal with test problems even if they forget a particular process or formula. Your student’s teacher also has access to a wide range of support materials that will help him or her to respond to an individual student’s pace, language issues, and need for additional assistance or enrichment.
Begin by taking stock of how your student uses his or her after-school time. Evaluate whether there is a suitable place with good light to make homework a comfortable activity, and whether distractions in the homework environment are manageable. Your support and praise are as important to your student’s success as the teacher’s guidance and the quality of the learning materials. You’ll want to make your support effort as thoughtful as possible.

Your Own Experience with Math Is a Big Influence

Did you do well in math when you were in school? If math was hard for you, you might actually find it easier to help your student than if it came easily to you, because you’ll be especially sympathetic. You’ve probably also developed some practical understanding since you left school. The important thing is to work hard to keep from passing on negative ideas about math. You have the chance to help your student have a better attitude toward math. Your message must be “mathematics is important for everyone.” To be successful in our society, everyone must be able to recognize when a situation needs a mathematical solution, to tell what quantities are involved, and to understand how to work toward a solution. Your student has the benefit of a better approach and better materials than you probably had.

What if you’re good at math? You will have to work hard to keep from dominating your student’s learning. It’s sometimes very hard to resist explaining an idea or giving an answer you understand, but holding back is necessary if your student is to remember the idea and ultimately become an independent learner. Praise all your student’s honest efforts and support his or her attempts to explain, question, or break down the problem.

No matter how comfortable you are with math, you can help your student reach the goals of the Discovering Algebra approach and learn algebra. Try to establish two habits when you work with your student.

First, be a student to your student. Keep asking him or her for explanations. Ask questions as if you were the student trying to learn. No matter how well you understand things yourself, asking “Why does that work?” is better than saying, “Here’s how to do that.”

Second, be curious and enthusiastic. Offer comments like, “I haven’t seen this idea before, but it seems interesting” rather than “It’s beyond me!” or “This isn’t important.” Ask what happened in class, ask what your student contributed and how well he or she understood, and be curious about the homework. Showing this kind of interest says that you expect your student to be actively involved in class and to work on homework every day.

Learn About and Use Other Resources

Use this guide in conjunction with the Discovering Algebra textbook. Refer to the notes on individual chapters. References are made to specific examples and exercises in the text. By all means, be aware of resources your student has at school and what he or she can access from home.
Use Tried and True Strategies

Some classic problem-solving strategies can help your student, and you can assist him or her to use them.

1. **Make an organized list.** Stating the facts given in a problem one at a time is especially helpful for a student with low reading skills, an attention deficit disorder, or a simple case of impatience. Be sure your student understands what is being asked for. Have your student do the writing. Another approach is to build a table of values, prices, or corresponding numbers. This will help your student **look for patterns** that give a clue to the answer or that lead to a solution process. Flow charts can also help a student work step by step.

2. **Draw a diagram.** This is very helpful for real-world problems, or problems that have geometric figures or a coordinate grid. Be sure your student is the one doing the drawing. You can coach, ask questions, or make suggestions: “Why not draw a line for the wall?” “Where is the person standing?” Encourage your student to **label parts of the diagram** with quantities that represent distance or other measurements, and use arrows for motion. Use stick figures or smiley faces to represent people.

3. **Eliminate possibilities.** Deciding what kind of solution is improbable or not at all possible can spark your student’s thinking process. If the “ingredients” for a process he or she wants to apply are not present in the problem, or if not enough information is available, a different line of reasoning is necessary.

4. **Solve an easier, related problem.** Substitute easier numbers in the given problem to make the process clear. Then put the “harder” or “messier” numbers back in and apply the same process. Or, just **work on one stage of a problem.** This might help your student recognize a process that he or she remembers and understands. It also reestablishes a climate of success. Be sure to praise success on easier problems or success with partial answers to demonstrate your support and to prove to your student that he or she has some level of ability and achievement.

5. **Work backward.** Start at the end of a series of steps and see how it feels to work toward the beginning. This is a good way to check whether a guessed answer is right and to understand why it was a good guess. **Guess-and-check** is a good strategy in itself if the student gets closer and closer to the right answer in successive steps.

If your student continues to have homework trouble even when you have tried to help, you can guide him or her to list questions for the teacher. This list will help the teacher know whether the student feels that he or she basically understood the lesson and is simply stuck on a single problem, or whether the student feels completely off track and hasn’t understood the lesson or even the last several lessons. Are there particular symbols that your student doesn’t understand? Is there an example in the book that he or she cannot follow? Helping your student script questions to the teacher will reduce anxiety or shyness about asking for help. If, finally, your student feels unable to ask for assistance, you should intervene with a note or call to the teacher.

Your student will be using a graphing calculator in class. You might want to purchase a graphing calculator for your student at home. Ask your student’s teacher for advice about the kind of calculator to purchase and where to purchase it. Unless the teacher recommends another calculator, a good choice is the TI-83 Plus or TI-84 Plus manufactured by Texas Instruments.
If you have Internet access, you can enrich your student’s experience by having your student follow Web links and view the Dynamic Algebra Explorations available for *Discovering Algebra*. You can also download Calculator Notes, Condensed Lessons, and More Practice Your Skills worksheets. Find these at www.keymath.com/DA.

If the teacher has registered, you can access the online version of *Discovering Algebra*. Discovering Algebra Online is a service that provides students with access to all the content of their printed textbook page by page, in an easy-to-use format. The online textbook has an interactive glossary and an index, and direct links to the chapter-specific resources just mentioned.

*Discovering Algebra* has been designed with an investigative approach to engage your student in doing mathematics—understanding, learning, remembering, and applying algebra skills. With your student’s growing sense of responsibility for his or her own learning, a teacher’s professional guidance, and your earnest support, your student will make gains in mathematics and have a positive experience with algebra.
Overview of Topics in Discovering Algebra

The arrangement of topics in Discovering Algebra: An Investigative Approach is carefully planned to help students develop connections between new and previously learned material and to build their understanding.

In Chapter 0, students review and increase their facility with some arithmetic skills by using them to solve problems. Students are introduced to the idea of recursion, an intuitive procedure of doing a process over and over again, each time building on the last step. Recursion will be used throughout the course.

In Chapters 1 through 5, students learn to set up and solve linear equations—equations whose graphs are lines—which is the heart of a beginning algebra course.

- In Chapter 1, students use graphs and statistical measures to organize and make sense of data.
- In Chapter 2, students work on proportional reasoning and learn how to solve equations by undoing, a powerful method that works for many types of equations.
- Chapter 3 allows the idea of linear expressions to grow out of proportional reasoning. Students are also introduced to the balancing method for solving equations.
- In Chapters 4 and 5, students see linear equations in other contexts. In Chapter 4 students deepen their understanding of linear equations by fitting lines to data (building on ideas in Chapter 1). Chapter 5 focuses on expanding the ideas of linear equations through systems of these equations, and through the introduction of inequalities.

In Chapters 6 through 9, students study nonlinear growth.

- In Chapter 6, students learn about exponential growth and equations.
- Chapter 7 generalizes linear and exponential growth to the idea of function.
- Chapter 8 shows how graphs of functions can be transformed.
- In Chapter 9, students investigate relationships between quadratic functions and their graphs and equations.
- Chapter 10 introduces probability and counting techniques.
- Chapter 11 previews geometry.

Chapter Summaries

In chapter summaries, the chapter content is briefly summarized, and important new words are italicized. A summary problem is presented, along with question prompts that you can use to get your student thinking. The summary problem is a comprehensive problem that will give you and your student a lot to talk about. The question prompts are followed by sample answers. Review exercises and solutions are provided at the end of the material for each chapter.
Chapter 0

Fractions and Fractals

Content Summary
The topic of Chapter 0 is the investigation of patterns in fractal designs. Don’t be intimidated if you haven’t seen fractals before. The main purpose of Chapter 0 is to add visual interest and new ideas to the review of fractions, decimals, signed numbers, and exponents. Students’ mathematical skills will be stronger, and they will understand the ideas behind them better, after applying them to fractals. Chapter 0 also introduces the ideas of recursion, random processes, and evaluating algebraic expressions, which will be important throughout the study of algebra and in later mathematics courses.

Your student’s teacher might select lessons from Chapter 0 to review skills and ideas that the class needs to understand better as students begin their formal study of algebra. Students may come to this course with varying levels of preparedness, and Chapter 0 is provided to help even up the skills of students. Don’t be alarmed if the teacher decides that some lessons are not necessary to study in your student’s class. The teacher can also use the time he or she spends on Chapter 0 to help students adjust to Discovering Algebra’s teaching and learning methods: doing investigations, working in groups, and using graphing calculators. Meanwhile, you can use this time to help establish patterns in the way you’ll work with your student. Each chapter provides a summary problem for you to discuss with your student.

Here is a summary of the main new ideas from Chapter 0.

Recursion
Using an example is the easiest way to understand recursion:

The first odd number is 1; the second is 3; the third is 5; and so on. You can find the fifteenth odd number by multiplying 2 by 15 (to get 30) and then subtracting 1. Or, you can find the fifteenth odd number the long way by counting 1, 3, 5, 7, 9, and so on, until you get up to 29.

Finding the fifteenth odd number by adding 2 repeatedly, starting from 1, is called recursion, because you find the result at each step from the result at the previous step. With technology, you can find recursive results very quickly. The recursive methods students see in Chapter 0 will be helpful as they study linear growth in Chapters 2 and 3 and exponential growth in Chapter 6.

| 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 | 29 |

Random Processes and Graphing Calculators
A process is random if individual results are unpredictable. Common examples of random processes are flipping coins and rolling dice.

Graphing calculators can generate random numbers. If you don’t have dice, you can have the calculator display a random whole number between 1 and 6 to simulate rolling a die. Or you can simulate flipping a coin by having the calculator pick either a 1 or a 2 at random. If a class of 30 students “counts off,” the teacher can randomly
select a class member by having the calculator display a random whole number between 1 and 30. But calculators are more flexible than that. For example, they can also display a random whole number between 1 and 100 to simulate picking one person out of a hundred at random.

**Evaluating Expressions**

Your student may already know that a mathematical expression is a combination of numbers and letters that together represent a single number. To evaluate an expression, you replace each letter with a number. For example, $3t + 5a - 4$ is an expression, which means $(3 \times t) + (5 \times a) - 4$. If you replace $t$ with 4 and $a$ with 1, the expression becomes $(3 \times 4) + (5 \times 1) - 4$. This is equivalent to the number 13, which is itself an expression. Some people mistakenly use the word equation to refer to an expression. An equation has two expressions, one on each side of an equal sign (=). Expressions themselves do not include equal signs and are not equations.

**Summary Problem**

You and your student might discuss the example on pages 15 and 16 in the Discovering Algebra textbook and use the Dynamic Algebra Exploration at www.keymath.com/DA to further explore the hat curve.

What recursive procedure will generate the last column of the table shown?

<table>
<thead>
<tr>
<th>Stage number</th>
<th>Number of segments</th>
<th>Length of each segment</th>
<th>Fraction form</th>
<th>Decimal form</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$1 \cdot 1$</td>
<td>1.00</td>
</tr>
<tr>
<td>1</td>
<td>$1 \cdot 5 = 5^1$</td>
<td>$1 \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^1$</td>
<td>$5^1 \cdot \left(\frac{1}{3}\right)^1 = \left(\frac{5}{3}\right)^1$</td>
<td>1.67</td>
</tr>
<tr>
<td>2</td>
<td>$5 \cdot 5 = 5^2$</td>
<td>$\frac{1}{3} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^2$</td>
<td>$5^2 \cdot \left(\frac{1}{3}\right)^2 = \left(\frac{5}{3}\right)^2$</td>
<td>2.78</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>17</td>
<td>$5^{17}$</td>
<td>$\left(\frac{1}{3}\right)^{17}$</td>
<td>$5^{17} \cdot \left(\frac{1}{3}\right)^{17} = \left(\frac{5}{3}\right)^{17}$</td>
<td>5907.84</td>
</tr>
</tbody>
</table>

Discuss these questions with your student from the point of view of a student to your student:

- Which column does the problem refer to?
- What does recursive rule or procedure mean?
- What might a starting value be for this column?
- How might you get from one row to the next in the table?
Chapter 0 • Fractions and Fractals (continued)

- What recursive rule captures that change?
- If you enter that recursive rule into your graphing calculator, do you get the same values as those shown in the table?
- Can you make the calculator give the answer in a fraction form as well as in a decimal form? How?
- Does your recursive rule reflect how the fractal was made? What is the connection?
- What if the original segment at Stage 0 were just \( \frac{2}{3} \) as long? Would that change the table and the formula?
- What other changes in the original problem might you explore?

Some of these questions have several possible valid answers. It is not important that you know all the answers. Instead, as you talk about the answers, make sure your student gives a good explanation of why an answer is reasonable. In particular, an answer of “yes” or “no” is not enough. Encourage your student to ask questions, too.

Sample Answer
The questions refer to the columns about the total length of the fractal at different stages. A recursive procedure gives the first stage and describes how to get from one stage to the next. Here, to get from the length of one stage to the length of the next, multiply by 5 and divide by 3, or multiply by \( \frac{5}{3} \). Multiplication by 5 reflects the fact that each segment is replaced with five segments; dividing by 3 reflects the fact that each of the new segments is \( \frac{1}{3} \) as long as each of the old segments.

If the fractal started off shorter but kept the same rule for getting to the next stage, the total length at each stage would change, but each stage would still be \( \frac{5}{3} \) the length of the previous stage.

Calculator Note 0A gives information about working with fractions on a calculator.

Encourage your student to be imaginative as he or she suggests other ways to change the original problem.
Chapter 0 • Review Exercises

1. *(Lesson 0.1)* Use this fractal pattern to answer the questions. Assume that the area of the Stage 0 square is 1.

   ![Stage 0, Stage 1, Stage 2 images]

   a. Describe the pattern, and draw Stage 3.
   b. What is the area of the smallest square at Stage 3? Write it in exponent form.
   c. What is the total area of the unshaded squares at Stage 2? At Stage 3?

2. *(Lessons 0.1, 0.2)* Do the following calculations. Leave your answers in fraction form.
   a. \( \frac{2}{5} + \frac{3}{20} \)
   b. \( 1 - \left( \frac{1}{3} + \frac{5}{12} \right) \)
   c. \( \frac{2}{3} \cdot \frac{1}{5} \)
   d. \( \frac{2}{3} + \frac{1}{2} - \left( \frac{1}{6} + \frac{1}{9} \right) \)

3. *(Lesson 0.2)* Rewrite the expression \( 3^5 \) as a repeated multiplication, and find the value.

4. *(Lesson 0.4)* Do the following calculations. Check your results on your calculator.
   a. \( 2 \cdot -4 + 1 \)
   b. \( 3 - 5 \cdot (1 - 4) \)
   c. \( 6 \div -2 \cdot 4 \)
   d. \( 2 - (2 \cdot 3 - 2) \)
SOLUTIONS TO CHAPTER 0 REVIEW EXERCISES

1. a. To create Stage 3, divide each white square in Stage 2 into four squares and shade the upper right square.

![Stage 3 diagram]

b. In each stage, the area of the smallest square is \( \frac{1}{4} \) times the area of the smallest square in the previous stage. In Stage 1, the area of the smallest square is \( \frac{1}{4} \); in Stage 2, it is \( \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \); and in Stage 3, it is \( \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64} \).

c. It helps to look for a pattern in the earlier stages. The area of the unshaded region in Stage 0 is 1. The area of the unshaded region in Stage 1 is \( \frac{3}{4} \). In Stage 2, the area of the unshaded region is \( \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16} \). From this pattern, the area of the unshaded region in each stage is \( \frac{3}{4} \) times the area of the unshaded region in the previous stage. In Stage 3, the area is \( \left(\frac{3}{4}\right)^3 = \frac{27}{64} \).

2. a. \( \frac{2}{5} \cdot \frac{4}{9} + \frac{3}{20} \) Multiply to get a common denominator.

\[
\frac{8}{20} + \frac{3}{20} = \frac{11}{20}
\]

Multiply.

Add numerators.

b. \( 1 - \left( \frac{9}{12} \right) \) Do the calculations in the parentheses first.

\[ \frac{1}{4} \]

Subtract and reduce.

c. \( \frac{2 \cdot 1}{3 \cdot 5} \) Multiply numerators and denominators.

\[ \frac{2}{15} \]

Multiply.

d. \( \frac{2}{3} + \frac{1}{2} - \left( \frac{3}{18} + \frac{2}{18} \right) \) Find a common denominator.

\[
\frac{2}{3} + \frac{1}{2} - \left( \frac{5}{18} \right)
\]

Add.

\[
\frac{12}{18} + \frac{9}{18} - \frac{5}{18}
\]

Find a common denominator.

\[
\frac{16}{18} = \frac{8}{9}
\]

Simplify the expression and reduce the fraction.

3. \( 3 \cdot 3 \cdot 3 \cdot 3 = 243 \)

4. a. \( 2 \cdot -4 + 1 = -8 + 1 = -7 \)

b. \( 3 - 5 \cdot (1 - 4) = 3 - 5 \cdot -3 = 3 - (-15) = 18 \)

c. \( 6 + 2 \cdot 4 = -3 \cdot 4 = -12 \)

d. \( 2 - (2 \cdot 3 - 2) = 2 - (6 - 2) = 2 - 4 = -2 \)
Data Exploration

Content Summary
In Chapter 1, students get used to organizing and analyzing information. Many problem situations in Discovering Algebra are realistic. Representing those situations using mathematical expressions, equations, graphs, or tables is called mathematical modeling.

In this chapter, students think about collections of numbers as single entities, which helps them understand the idea of an algebraic variable more deeply. In Lesson 1.6, the chapter makes a transition from one-variable information to relations involving two-variable information.

Data are pieces of information, which may or may not be numbers. In this chapter, a collection of numerical data, also called a data set, is represented on the calculator as a single entity called a list or matrix. There are several ways to summarize a collection of data, or data set.

Statistics
Some data set summaries use numbers, called statistics. The most familiar statistic is the mean, often called the average. It’s one way of describing the center of the data set. However, other statistics, such as the median and mode, are sometimes more useful than the mean for describing data.

Often, you want to know more about a data set than any center statistic, or measure of center, will tell you. The range and quartiles are statistics that help describe how the data set is spread out.

Statistical Graphs
Statistical graphs are pictorial representations of data sets.

Sometimes a data set gives numbers of items that fall into various non-numerical categories. Pictographs and bar graphs give pictures of these data sets, as shown on pages 39, 40, and 42 of your student’s textbook.

If the categories themselves are individual numbers or intervals (ranges) of numbers, the most common graphs are histograms, dot plots, stem-and-leaf plots, and box-and-whiskers plots. See examples on pages 41, 53, 59, 61, and 63.

(continued)
Chapter 1 • Data Exploration (continued)

Many data sets contain pairs of numbers. These pairs are usually graphed in a scatter plot. Students draw straight lines, or trend lines, through some scatter plots to help make predictions, as on page 79. (This also connects to linear equations in Chapter 3.) In the Guesstimating Investigation, on pages 77 and 78, students use the equation \( y = x \) to model data points that have the form (actual, estimate). The equation \( y = x \) is the parent of all linear equations.

Matrices
Another way to represent data sets of paired numbers or other information is in a table. The mathematical form of a table is a matrix. Matrices are especially useful when data sets are being combined. Matrices will be used again for solving systems of linear equations in Chapter 5 and for representing transformations in Chapter 8. If your state standards do not include matrices, your student’s teacher may skip the lessons on matrices.

Summary Problem
You and your student might discuss this summary problem from Chapter 1. It’s a good problem to revisit several times while working through the chapter.

What statistics and graphs could be used to describe the Jurassic Park data from page 77?

Questions you might ask in your role as a student to your student include:

- What measures of center might you use for either the actual data or the estimated data?
- What measures of spread might you use for either of these columns?
- What graphs might you use for either of these columns?
- How can you use your graphing calculator to measure any of these statistics or to create a graph to compare the two columns?
- What observations can you make when comparing the estimated and actual numbers?
- What happens to the various statistics and graphs if you change the data slightly?

Encourage your student to ask other questions.

Sample Answers
The center of a data set can be measured using mean, median, and mode. Its spread can be measured using range, minimum, maximum, quartiles, interquartile range, and outliers. For these data, the mean (19.5 actual, 15.8 estimated) or median (17 or 18) would be good measures of center.

A bar graph or histogram would clearly show the center and the spread of one column. A pair of bar graphs is good for comparing two data sets, such as the two columns.

The Calculator Notes for Chapter 1 describe the calculator graphs, including the scatter plot, that can show both actual and estimated numbers.

Some observations: The estimates were generally close to the actual numbers. The velociraptors and procompsognathids were greatly underestimated.

To help your student think about what happens to the statistics when data are changed, you might ask about removing outliers. (How does the mean change? The median? Which changes more?) You also might explore this idea using the Dynamic Algebra Explorations Finding the Center and The Box Plotter at www.keymath.com.
Chapter 1 • Review Exercises

Name ___________________________ Period _____________ Date ________________

1. (Lessons 1.1, 1.2) Some students at a class picnic gathered data on how many grapes they each had on their plate. The results are shown here on a dot plot.

![Dot plot of grape data](image)

a. How many students had more than six grapes on their plate?
b. What is the range of the data?
c. Find the mean, median, and mode of the data.

2. (Lessons 1.3, 1.4) The scores on a midterm in a history class were 98, 65, 77, 83, 77, 79, 92, 84, 71, 73, 90, 77, 92, 64, and 83.

a. Create a stem-and-leaf plot for the data.
b. Create a histogram for the midterm scores.
c. Give the five-number summary for the midterm scores, and draw a box plot.
d. What is the interquartile range (IQR) of the midterm scores?

3. (Lessons 1.6, 1.7) Annie and Sonya went grocery shopping, and they estimated the cost of each item on their shopping list before purchasing it. The table below shows their estimated cost and the actual cost for each item.

<table>
<thead>
<tr>
<th>Item</th>
<th>Estimated cost</th>
<th>Actual cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eggs</td>
<td>$1.50</td>
<td>$2.19</td>
</tr>
<tr>
<td>Bread</td>
<td>$2.00</td>
<td>$2.69</td>
</tr>
<tr>
<td>Cheese</td>
<td>$5.50</td>
<td>$3.57</td>
</tr>
<tr>
<td>Tea</td>
<td>$4.00</td>
<td>$2.60</td>
</tr>
<tr>
<td>Napkins</td>
<td>$2.50</td>
<td>$4.29</td>
</tr>
<tr>
<td>Ketchup</td>
<td>$3.00</td>
<td>$1.59</td>
</tr>
<tr>
<td>Oil</td>
<td>$1.50</td>
<td>$1.69</td>
</tr>
<tr>
<td>Vinegar</td>
<td>$2.50</td>
<td>$1.59</td>
</tr>
<tr>
<td>Lettuce</td>
<td>$0.50</td>
<td>$0.99</td>
</tr>
</tbody>
</table>

a. Construct a scatter plot of \((\text{estimated cost}, \text{actual cost})\). Label each point with an appropriate abbreviation. Also construct the \(y = x\) line.
b. Which items lie above the \(y = x\) line? What does this mean?
c. Which items lie below the \(y = x\) line? What does this mean?
1. a. Add up the number of dots above every number greater than six; 6 + 2 + 1 = 9 students.
   
   b. 12 − 0 = 12.
   
   c. Mean:
   
   \[
   \frac{3(0) + 1(2) + 4(3) + 3(4) + 4(5) + 1(6) + 6(7) + 2(8) + 1(12)}{3 + 1 + 4 + 3 + 4 + 1 + 6 + 2 + 1} = 4.88
   \]
   
   median: 5; mode: 7.
   
2. Start by writing the data in order: 41, 64, 65, 71, 73, 77, 77, 79, 83, 83, 84, 90, 92, 92, 98.
   
   a. Put the tens digits in the “stem,” and the ones digits in the “leaves.” Create a key that shows how to interpret one of the entries.
   
   4 1
   
   5
   
   6 45
   
   7 137779
   
   8 334
   
   9 0228
   
   Key 7 1 means 71
   
   b. b. Start with the data in ascending order. The median is the middle value. If there are an even number of data points, the median is the average of the two middle values. Q1 is the median of all the data below the median, and Q3 is the median of all the data above the median. The five-number summary is minimum, Q1, median, Q3, maximum.
   
   Therefore, the five-number summary is 41, 72, 78, 87, 98.
   
   d. IQR = Q3 − Q1 = 87 − 72 = 15
   
3. a. b. Lettuce, oil, eggs, bread, and napkins; these are the items whose actual cost was higher than the estimated cost.
   
   c. Vinegar, ketchup, tea, cheese; these are the items whose actual cost was lower than the estimated cost.
CHAPTER 2

Proportional Reasoning and Variation

Many real-life problems have either no answer or more than one valid answer. In *Discovering Algebra*, students see some exercises with more than one answer. For example, in Lesson 2.1, Exercise 7 instructs, “Write three other true proportions using these four integers.” Continue to focus on having your student explain his or her reasoning rather than on getting the “right answer.” If the reasoning is good, then good answers will follow.

In Chapter 2, students get more comfortable with solving problems that they haven’t been told how to solve. For example, in the Just Undo It! Investigation of Lesson 2.8, students are asked to explain a number trick rather than being given an explanation. This approach builds skills and confidence in solving problems.

Content Summary

Chapter 2 continues the modeling journey. Students focus on writing equations and solving equations for an unknown number. Chapter 2 equations are proportions, which involve ratios. Students learn about direct and indirect variation and learn to solve equations by undoing operations.

Ratios

A ratio is a comparison of one quantity relative to another by division, so *Discovering Algebra* shows most ratios as fractions. Students might already think of fractions as parts of a whole (2 of 3 equal parts), as implied division (2 divided by 3), or as numbers (a value between 0.5 and 1). Thinking about ratios can be challenging. In fact, when working mathematicians disagree about something, such as a question of probability, a ratio is often involved.

Proportions

A proportion is an equation stating that two ratios are equal. Four quantities are involved in a proportion, two in each ratio. When one of these quantities is unknown, you can solve the proportion to find it. In their book, students see how proportions can represent various situations, including conversion of measurement units and capture-recapture studies.

If you solved proportions in school, you might have used a method called “cross-multiplying.” It’s an efficient method when used correctly, but students tend to latch on to it without understanding why it works, and then often use it in situations where it doesn’t apply. To build understanding and reinforce thinking, *Discovering Algebra* avoids cross-multiplication. You can solve all proportion equations at this level by multiplying to undo the division, perhaps after inverting the proportion. See the Multiply and Conquer Investigation on pages 97 and 98.

Dimensional Analysis

Making conversions between units in different systems of measurement is an important skill for school and for daily life. For example, you might need to compare metric units with English units, fluid measurements with cup measurements, or miles per hour with feet per second. Dimensional analysis is a strategy that leaves no doubt about whether you should multiply or divide by a conversion factor. See Example B on pages 109 and 110.

(continued)
Chapter 2 • Proportional Reasoning and Variation (continued)

Direct Variation
Studying direct variation is an excellent way to deepen students’ understanding of linear relationships and to prepare them for further study of linear equations in Chapters 3 through 5.

A direct variation is an equation such as \( y = \frac{3}{2}x \) or \( y = -5x \). These equations are equivalent to the proportions \( \frac{y}{x} = \frac{3}{2} \) and \( \frac{y}{x} = -\frac{5}{1} \). In general, a direct variation is an equation of the form \( y = kx \), where \( k \) is a number.

If the constant (unchanging) number \( k \) is considered as a ratio with denominator 1, this number is called a rate. Many conversion factors, such as \( \frac{5280 \text{ feet}}{1 \text{ mile}} \), are rates. So are speeds, in \( \frac{\text{miles}}{\text{hour}} \) or \( \frac{\text{feet}}{\text{second}} \). When speed (rate) is constant, the equation relating distance and time traveled is a direct variation. A graph of \((\text{time, distance})\) of an object moving at a constant speed is a line.

Inverse Variation
Chapter 2 also considers inverse variation, although an inverse variation is not a linear equation. Rather, an inverse variation is an equation of the form \( y = \frac{k}{x} \), where \( k \) is a constant number. These equations and their graphs will reappear in Chapter 8.

Graphs of Variations
Variations are equations, so they have graphs. Students see the graphs of both direct and indirect variations in this chapter, although only the graphs of direct variations are straight lines. Students learn that \( k \), the constant of variation, is related to the steepness of the line. To keep students focused on the rate-of-change concept, the explanation for the mathematical term slope is delayed until Chapter 4.

Summary Problem
You and your student might discuss the following problem. It’s a good problem to revisit several times while working through the chapter.

What problem situations can you represent by the proportion \( \frac{x}{150} = \frac{17}{20} \), or by an inversion of one or both sides of the proportion?

Questions you might ask in your role as student to your student include:

- Could this proportion represent a capture-recapture problem?
- Can you change the proportion to represent a percent problem?
- Can you change the proportion to represent a problem about measurement?
- Can you present several methods for solving the problem?

Sample Answer
Many different answers are possible. Encourage students to be creative and to discuss the interpretation of each number in the proportion by using different contexts. The proportion might represent a capture-recapture problem. In a sample of 20 fish, 17 were found to be tagged. If a scientist predicted that 150 fish total are in the lake, how many fish did the scientist tag? To represent a percent problem, the ratio \( \frac{17}{20} \) might be interpreted as a quiz score, whereas \( \frac{17}{150} \) is the equivalent score on the final exam. The ratio \( \frac{17}{20} \), or 85%, is equivalent to both scores. For measurement, the scale on a map might show that 20 cm represents 150 km. If two cities are 17 cm apart on the map, how many km apart are they?

Remind students to multiply by 150 to undo the division by 150.
Chapter 2 • Review Exercises

1. *(Lessons 2.1, 2.2)* Write a proportion to answer each question. Then solve the proportion.
   a. What number is 30% of 75?
   b. What percent of 16 is 125?
   c. 13 is 0.5% of what number?

2. *(Lesson 2.3)* One kilogram (kg) is approximately 2.2 pounds (lb). Which is heavier—a 17 lb object or an 8 kg object?

3. *(Lessons 2.4, 2.5)* Determine whether each relationship describes direct or inverse variation. In each case, write an equation in \( y = \) form, and then graph it on your calculator.
   a. \( xy = 0.5 \)
   b. \( y = 0.5x \)
   c. The relationship between the number of pages of a book Ari reads and the amount of time it takes him to read it, if he reads at a constant rate of 2 minutes per page.
   d. The relationship between the speed a car travels and the amount of time it takes to cover a distance of 30 mi.

4. *(Lessons 2.7, 2.8)* Identify the order of operations. Then create an undo table and use it to solve the equation.
   \[
   \frac{2x - 1}{3} + 5 = 8
   \]

5. *(Lesson 2.8)* A car rental company charges a flat rate of $30, plus $0.20 for every mile driven.
   a. Write an equation that relates the cost of renting a car to the number of miles driven.
   b. Gloria rented a car from this company for a weekend trip, and her bill came to $58.40. How many miles did Gloria drive?
1. a. \( x = \frac{30}{100} \) The proportion.
\[ x = \frac{30}{100} \cdot 75 \] Multiply by 75 to undo the division.
\[ x = 22.5 \] Multiply and divide.
22.5 is 30\% of 75.

b. \( x = \frac{125}{16} \) The proportion.
\[ x = \frac{125}{16} \cdot 100 \] Multiply by 100 to undo the division.
\[ x = 781.25 \] Multiply and divide.
125 is 80\% of 16.

c. \( x = \frac{0.5}{100} \) The proportion.
\[ x = \frac{100}{0.5} \] Invert the proportion.
\[ x = \frac{100}{0.5} \cdot 13 \] Multiply by 13 to undo the division.
\[ x = 2600 \] Multiply and divide.
13 is 0.5\% of 2600.

2. Use proportions to convert both weights to the same units. For example, convert 8 kilograms to pounds. Let \( x \) be the weight of the 8 kg object, in lb.
\[ \frac{2.2}{1} = \frac{x}{8} \] The proportion.
\[ (8 \text{ kg}) \cdot \frac{2.2}{1} = x \] Multiply by 8 to undo the division.
\[ x = 17.6 \] Multiply and divide.
The 8 kg object is heavier.

3. a. Inverse variation; \( y = \frac{0.5}{x} \).

b. Direct variation; \( y = 0.5x \).

c. Direct variation; \( y = 2x \), where \( x \) is the number of pages Ari reads, and \( y \) is the number of minutes he takes to read them.

\[ [0, 40, 5, 0, 30, 5] \]

d. Inverse variation; \( y = \frac{30}{x} \), where \( x \) is the speed of the car in mi/h, and \( y \) is the number of hours it takes the car to cover 30 mi.

\[ [0, 90, 10, 0, 5, 1] \]

4. \( x = 5 \)

\begin{tabular}{|c|c|c|}
\hline
\textbf{Description} & \textbf{Undo} & \textbf{Result} \\
\hline
Pick \( x \) & & 5 \\
\times (2) & \div (2) & 10 \\
\div (1) & + (1) & 9 \\
\div (3) & \times (3) & 3 \\
\div (5) & - (5) & 8 \\
\hline
\end{tabular}

5. a. \( y = 30 + 0.2x \), where \( x \) is the number of miles driven and \( y \) is the cost of the car rental.

b. 142 mi

\begin{tabular}{|c|c|c|}
\hline
\textbf{Description} & \textbf{Undo} & \textbf{Result} \\
\hline
Pick \( x \) & & 142 \\
\times (0.2) & \div (0.2) & 28.40 \\
\div (30) & - (30) & 58.40 \\
\hline
\end{tabular}
At this point, you might think about how you and your student are interacting. For example, are you being a student to your student? Do you explain as little as possible or just enough so that your student is becoming an independent learner and thinker? Is the pencil or calculator in your student’s hands? Do you try not to answer questions that your student hasn’t asked? Telling your student too much can waste time, because your student might not understand. It can lead to your student feeling overwhelmed. Deeper understanding can result when you allow your student to teach the concept or skill to you and others.

Content Summary
Chapter 3 focuses on equations of lines. Students expand their idea of linearity, and they learn to work backward.

Linearity
You can think of linearity in several ways. Linearity as constant rate of change. One way to think about linearity is that the rate of change of one variable in relation to the other is constant. You start somewhere and advance by the same amount at each step. This kind of change is called linear growth, although the values will be shrinking instead of growing when the rate of change is negative. With variation in Chapter 2, growth always started at 0; in this chapter, growth can start with any value.

Linearity as equations of the form $y = a + bx$. Another way of thinking about linearity is through equations that relate variables. In this book, linear equations have the intercept form $y = a + bx$. This form indicates the starting point, $a$, and what’s added to it, $b$, each time $x$ increases by one unit. The traditional slope-intercept form $y = mx + b$ that you might remember is mentioned in Chapter 4, but the intercept form introduced in this chapter better reflects the idea of growing at a constant rate from a starting point.

Linearity as graphs of lines. You can also understand linearity through graphs. The equation $y = a + bx$ is graphed by starting at point $(0, a)$ and moving vertically by amount $b$ for each unit moved across from left to right.

Working Backward
Many real-life situations call for predicting when a quantity will grow to a certain value. Ways of making that prediction reflect the three ways of thinking about linearity. From the growth perspective, you might think of counting the steps as you repeatedly add on to the starting point until you reach the desired value. This can be done by hand, or you can use home-screen recursion or sequences on a graphing calculator.

If the situation is represented by an equation, there might be two methods of solving it: the undoing method and the balancing method. If you know that $3x + 2 = 17$, you can use the undoing method and think, “I multiply $x$ by 3 and add 2 to get 17. To find $x$, I can undo that process, beginning with 17. I subtract 2 (to get 15) and then divide by 3 (to get 5).” You’ll need the balancing method if the unknown appears more than once. Applying the balancing method to the equation $3x + 2 = 17$, you subtract 2 from both sides to get $3x = 15$, and then divide both sides by 3 to get $x = 5$. (continued)
Chapter 3 • Linear Equations (continued)

Summary Problem
You and your student might discuss this summary problem from Chapter 3. It’s a good problem to revisit several times while working through the chapter.

Here’s a table showing the heights above and below ground of different floor levels in a 25-story building (taken from page 158):

<table>
<thead>
<tr>
<th>Floor number</th>
<th>Basement (0)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>…</th>
<th>10</th>
<th>…</th>
<th>…</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (ft)</td>
<td>−4</td>
<td>9</td>
<td>22</td>
<td>35</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>217</td>
<td></td>
</tr>
</tbody>
</table>

What floor has a height of 282 feet?

Questions you might ask in your role as student to your student include:

- How far apart are the floors?
- What could the negative number mean?
- Could you solve this problem by recursion on a graphing calculator?
- Is it possible to represent the height by an algebraic expression?
- Does the distance between floors appear in the expression?
- Does the height of the basement appear in the expression?
- Can the whole problem be represented by an equation?
- Can you graph the equation?
- What are various ways of solving the equation?
- Could you make an equation that tells how long it takes an elevator to reach various floors?
- In the Empire State Building in New York City, the floors vary in heights. Could you still write an equation that might be useful for either the heights or the elevator time?

Sample Answer
The floors, which start with a negative number (possibly meaning that the basement floor is below ground level), are 13 feet apart. To solve the problem on a graphing calculator, you could start with −4 and repeatedly add 13 until you reach 282. It would be most efficient to use recursion on a list [see Calculator Note 3A] to keep track of both the floor number and the height.

Or you can solve the equation \( \text{height} = -4 + 13 \cdot (\text{floor number}) \), or \( 282 = -4 + 13x \). Using the balancing or the undoing method, you can solve the equation to get \( x = 22 \).

Ask what 22 represents (the floor with a height of 282 feet). If you know the time it takes for the elevator to travel one floor, you can use that number in place of 13 to find the time it takes the elevator to travel from the basement to any other floor.

For buildings with irregular floor heights, you might use an equation containing the average height to make estimates of height or time. Or you might use different equations for different parts of the building.
Chapter 3 • Review Exercises

Name ___________________________ Period ________ Date __________

1. (Lessons 3.1, 3.2) Plot the first six points represented by each recursive routine.
   a. \([-4, 2]\) \[\{\text{Ans}(1) + 1, \text{Ans}(2) + 3\}\]
   b. \([0, 1.5]\) \[\{\text{Ans}(1) + 1, \text{Ans}(2) - 0.25\}\]
   c. \([2, -2]\) \[\{\text{Ans}(1) + 1, \text{Ans}(2) + 0.5\}\]

2. (Lessons 3.3, 3.4) The table at right shows a person’s distance from a motion sensor at various times.
   a. Describe the walk shown in the table. Include where the walker started, how quickly the walker walked, and in what direction the walker walked.
   b. Write a linear equation for the walk, in intercept form. Graph the equation and plot the points from the table.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Distance (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>2</td>
<td>1.1</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
</tr>
<tr>
<td>4</td>
<td>1.9</td>
</tr>
</tbody>
</table>

3. (Lessons 3.5, 3.6) A local theater company has a yearly membership fee, and members pay a reduced per-ticket cost. The equation \(C = 25 + 8n\) expresses the total cost \(C\) of purchasing \(n\) tickets in a single year.
   a. Based on the equation above, what is the yearly membership fee? What is the cost per ticket?
   b. If a person does not want to buy a membership, theater tickets cost $10 each. Write an equation for the total cost \(C\) of purchasing \(n\) tickets for someone without a membership.
   c. Graph both equations for the cost of \(n\) tickets. What is the rate of change of the cost for a member? For a non-member?
   d. Christina looked at the schedule for the upcoming theater year, and she found 12 shows that she would like to attend. Should she buy a membership?
   e. Last year, Christina bought a membership, and she spent a total of $137 that year on the membership fee and theater tickets. How many tickets did she buy?

4. (Lesson 3.6) Give the additive inverse of each number.
   a. 1          b. \(-1.25\)         c. \(\frac{3}{4}\)         d. \(\frac{6}{5}\)

5. (Lesson 3.6) Give the multiplicative inverse of each number in Exercise 4.
SOLUTIONS TO CHAPTER 3 REVIEW EXERCISES

1. a. The graph should include the points (−4, 2), (−3, 5), (−2, 8), (−1, 11), (0, 14), and (1, 17).
   ![Graph 1](image1)
   b. The graph should include the points (0, 1.5), (1, 1.25), (2, 1), (3, 0.75), (4, 0.5), and (5, 0.25).
   ![Graph 2](image2)
   c. The graph should include the points (2, −2), (3, −1.5), (4, −1), (5, −0.5), (6, 0), and (7, 0.5).
   ![Graph 3](image3)

2. a. The walker started 0.3 m away from the sensor, and walked away from the sensor at a rate of 0.4 m/s.
   b. $y = 0.3 + 0.4x$
   ![Graph 4](image4)

3. a. $25; 8$. The total cost is \((\text{membership fee}) + (\text{cost per ticket}) \times (\text{number of tickets})\).
   b. $C = 10n$
   c. Graph \(y = 25 + 8x\) and \(y = 10x\); Rate of change for member: $8$ per ticket; for non-member: $10$ per ticket.
   ![Graph 5](image5)
   d. Christina should not buy a membership. Compare the total cost under each option. With membership, \(C = 25 + 8 \times 12 = 25 + 96 = 121\); without membership, \(C = 10 \times 12 = 120\). She will save $1 by not getting a membership.
   e. 14 tickets. Solve the equation \(25 + 8n = 137\). This solution uses the balancing method; students might also solve by undoing.
   
   \[
   \begin{align*}
   25 + 8n &= 137 & \text{Original equation.} \\
   25 + 8n - 25 &= 137 - 25 & \text{Subtract 25 from both sides.} \\
   8n &= 112 & \text{Combine like terms.} \\
   n &= 14 & \text{Divide both sides by 8.}
   \end{align*}
   \]
   ![Graph 6](image6)

4. Take the opposite of each number. The additive inverse is the number that is added to the given number to equal 0.
   a. \(-1\)  
   b. \(1.25\)  
   c. \(-\frac{3}{4}\)  
   d. \(\frac{6}{5}\)

5. Find the reciprocal of each number. The multiplicative inverse is the number that is multiplied by the given number to equal 1.
   a. \(1\)  
   b. \(-0.8\)  
   c. \(\frac{4}{3}\)  
   d. \(-\frac{5}{6}\)
Learning involves what psychologists call *dissonance*: Unless people see that they don’t understand, they can’t really learn. Learning an idea involves adjusting your concepts so that the new idea fits. Because we want our students to understand, we might rush to explain concepts rather than let students come to realize on their own that they don’t understand. Try to decrease your student’s frustration with not understanding a concept right away. Show how your own curiosity allows you to welcome initial questions and confusion as an opportunity to learn.

**Content Summary**

Chapter 4 builds on the ideas of Chapter 3. So students may find that they don’t understand those earlier ideas as well as they thought they did. In this chapter, the notion of steepness gets formalized (more strictly defined), and your student learns how to derive another form of linear equations. Challenge your student to be patient as he or she works to understand how to make predictions from data points that appear somewhat linear but don’t all lie on any single line.

**Slope of a Line**

In Chapter 3, students learned that the equation $y = a + bx$ represents a line that goes through point $(0, a)$ and climbs $b$ units for each unit across. So, $b$ measures the steepness of the line. It may be difficult to measure how much a line climbs in one unit; it’s often easier to find how much it climbs over a larger number of units, and then divide. The result is the amount that it climbs for each unit. This number $b$ is the line’s *slope*, the mathematical term for *steepness*.

To calculate the slope of a line, students draw a *slope triangle* to find how much the line climbs over some convenient horizontal distance. From this idea comes the formula for the slope of a line through two points; the slope is the change in vertical distance divided by the change in horizontal distance.

**Point-Slope Form of a Linear Equation**

One reason the text emphasizes the intercept form of linear equations ($y = a + bx$) in Chapter 3 rather than the slope-intercept form ($y = mx + b$) is to convey the idea that linear growth starts at $a$ and climbs $b$ units for each unit change in $x$. Another reason is that, if the growth starts at values of $x$ other than 0, the equation generalizes naturally. For example, if the growth begins at point $(x_1, y_1)$, the equation is $y = y_1 + b(x - x_1)$. This is the point-slope form of a linear equation.

In Lesson 4.4, students use the distributive property to rewrite equations in point-slope form as equivalent equations in intercept form.

**Lines of Fit**

A major reason for studying equations of lines is to learn to make predictions. If you have several data points that lie on a line and you want to predict where another point $(x, y)$ will lie, you can find the equation of the line and evaluate it to find $y$ for a given $x$-value, or vice versa.

(continued)
Chapter 4 • Fitting a Line to Data (continued)

Most points in a real-world data set don’t lie on a single straight line, no matter how linear they look. Measurement error and other “field” factors come into play. So to make predictions, you need to find a line that comes close to the data points. This kind of line is called a line of fit for the data. Finding lines of fit gives your student a context to practice finding slopes and equations, and it has useful applications in science and business.

Finding a line of fit by adjusting a calculator graph until the line looks like a good fit gives students experience with relating a line’s steepness and \(y\)-intercept to its equation. Finding a line of fit by calculating the equation of the line through two particular data points gives students experience with using the slope formula and the point-slope form of a linear equation.

This chapter also shows students how to find a line of fit using Q-points, which builds on the statistics of Chapter 1. Later, in Discovering Advanced Algebra, students will learn the method of linear regression to find what’s often called the line of best fit.

Summary Problem

Here’s a table showing the federal minimum wage in various years (taken from Exercise 10 in the Chapter Review). Predict what the minimum wage will be in 2020.

<table>
<thead>
<tr>
<th>Year</th>
<th>1975</th>
<th>1980</th>
<th>1990</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wage ($)</td>
<td>2.00</td>
<td>3.10</td>
<td>3.80</td>
</tr>
</tbody>
</table>

Questions you might ask, in your role as student to your student, include:

- If you draw a line that fits the three points well, what would the line’s slope and \(y\)-intercept be? What prediction would you make?
- If you drew lines through two of the three points, what would the lines’ equations be? What predictions would they lead to?
- If you drew a line using Q-points from all the data in Exercise 10, what would the line’s equation be? What prediction would it lead to?
- Which, if any, of those lines do you think leads to the best prediction?

Sample Answers

Using different pairs of points will give different lines. For example, the equation for the line between (1975, 2.00) and (1990, 3.80) is approximately \(y = 2 + 0.12(x - 1975)\). That equation would predict a value of $7.40 for 2020. A line of fit for all three points might have approximately the same slope but a slightly higher \(y\)-intercept.

To find the equation for the line through the Q-points using all the data from p. 270, you find the quartiles of the \(x\)-coordinates and \(y\)-coordinates. The first and third quartiles of the \(x\)-coordinates are 1976.5 and 1990.5; those of the \(y\)-coordinates are about 2.25 and 4.03. So the Q-points that the line of fit will go through are (1976.5, 2.25) and (1990.5, 4.03). The equation for the line through those points has slope \(\frac{4.03 - 2.25}{1990.5 - 1976.5} = 0.13\). The equation \(y = 2.25 + 0.13(x - 1976.5)\) gives an estimate of $7.90 in 2020. Your student may have various reasons to prefer a particular line.
Chapter 4 • Review Exercises

Name _____________________________________ Period __________ Date ________________

1. (Lessons 4.1, 4.3) Consider the line passing through the points (2, 4) and (5, −0.5).
   a. Find the slope of the line.
   b. Use the slope to find two other points on the line.
   c. Write the equation of the line in point-slope form.

2. (Lesson 4.2) Sid went for a drive, and after he started he decided to use his trip meter to keep track of the distance he had traveled. He collected the data shown in the table.
   a. Plot the data on your calculator. Do the data look approximately linear?
   b. Look at your scatter plot, and choose two points that seem to be representative of the slope of the data. Find the slope of the line passing through those two points. What is the real-world meaning of the slope in this situation?
   c. Using the slope you found in 2b, adjust the y-intercept to find a line of fit for the data. What is the real-world meaning of the y-intercept in this situation?

3. (Lesson 4.4) Use the distributive property to write an equivalent equation in intercept form.
   a. \( y = 3 + 2(x + 1) \)
   b. \( y = 1 + 3(x - 5) \)
   c. \( y = -5 - (x - 8) \)

4. (Lesson 4.4) Factor each expression so that the coefficient of \( x \) is 1.
   a. \( 4x - 36 \)
   b. \( -2x + 10 \)
   c. \( -3x - 15 \)
   d. \( 2x + 7 \)

5. (Lessons 4.6, 4.7) Consider the following data set.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.0</th>
<th>1.7</th>
<th>2.3</th>
<th>3.2</th>
<th>3.5</th>
<th>4.1</th>
<th>4.9</th>
<th>6.2</th>
<th>7.1</th>
<th>7.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>35</td>
<td>31</td>
<td>31</td>
<td>28</td>
<td>27</td>
<td>24</td>
<td>25</td>
<td>17</td>
<td>20</td>
<td>18</td>
</tr>
</tbody>
</table>

   a. Find the five-number summaries for the \( x \)-values and the \( y \)-values.
   b. Create a scatter plot of the data, and determine which Q-points should be used to model this data.
   c. Find the line of fit based on Q-points for the data. Add the graph of this line to your graph for 5b.
1. a. \(-1.5.\) To find the slope, divide the difference in y-values by the difference in x-values.
\[
\frac{-0.5 - 4}{5 - 2} = \frac{-4.5}{3} = -1.5
\]
b. Answers will vary. Sample answer: Start with the point \((2, 4)\); add 1.5 to the y-value and subtract 1 from the x-value. Repeat this process with the new point you obtain. The resulting points are \((1, 5.5)\) and \((0, 7)\).
c. Answers will vary. Using the point \((2, 4)\), the equation is \(y = 4 - 1.5(x - 2)\). Using the point \((5, -0.5)\), the equation is \(y = -0.5 - 1.5(x - 5)\).
2. a. Yes, the data look approximately linear. See the solution for 2c.
b. Answers will vary. Using the first point and the last point, you get a slope of approximately 36.3. The slope is Sid’s average speed in mi/h.
c. Answers will vary. The line \(y = 52.2 + 36.3x\) appears to be a good fit. The y-intercept is an estimate for the reading on the trip meter when Sid started timing his trip.

![Graph](image)

\([0, 6, 1, 0, 280, 50]\)

3. a. \(y = 3 + 2(x) + 2(1)\) Use the distributive property.
\(y = 3 + 2x + 2\) Multiply.
\(y = 5 + 2x\) Add.
b. \(y = 1 + 3(x) + 3(-5)\) Use the distributive property.
\(y = 1 + 3x - 15\) Multiply.
\(y = -14 + 3x\) Add.
c. \(y = -5 - 1(x) - 1(-8)\) Use the distributive property.
\(y = -5 - x + 8\) Multiply.
\(y = 3 - x\) Add.

4. a. \(4x - 36 = 4(x) - 4(9)\) Factor 4 out of each term.
\(= 4(x - 9)\) Factor.
b. \(-2x + 10 = -2(x) + -2(-5)\) Factor -2 out of each term.
\(= -2(x - 5)\) Factor.
c. \(-3x - 15 = -3(x) + -3(5)\) Factor -3 out of each term.
\(= -3(x + 5)\) Factor.
d. \(2x + 7 = 2(x) + 2(3.5)\) Factor 2 out of each term.
\(= 2(x + 3.5)\) Factor.
5. a. 1.0, 2.3, 3.8, 6.2, 7.4; 17, 20, 26, 31, 35. See solution to Chapter 1 Exercise 2c in this guide for help in finding five-number summaries.
b. The Q-points you should use are \((2.3, 31)\) and \((6.2, 20)\). To find the Q-points, graph vertical lines extending from the Q1- and Q3-values for the x-values 2.3 and 6.2, and graph horizontal lines extending from the Q1- and Q3-values for the y-values 20 and 31. This will create a rectangular box. The Q-points are the corners of the box. Use the two Q-points that create the best line of fit for the data, in this case \((2.3, 31)\) and \((6.2, 20)\).

c. The slope of the line is \(\frac{-20 - 31}{6.2 - 2.3} = \frac{-11}{3.9} \approx -2.82\). Use the point-slope form of the equation, with the slope and either of the two Q-points found in 5b. The equation is \(y = 31 - 2.82(x - 2.3)\) or \(y = 20 - 2.82(x - 6.2)\).
CHAPTER 5

Systems of Equations and Inequalities

Continue to think about your interactions with your student regarding mathematics. Remember to encourage your student to become an independent learner and thinker by asking him or her to do the explaining. If you are explaining, you are not giving your student a chance to see what they don’t understand.

Content Summary
Chapter 5 reinforces ideas of linearity from Chapter 3. Chapter 5 presents problems modeled by more than one linear equation at a time and then considers problems represented by linear inequalities.

Systems of Linear Equations
Many real-world problems involve situations in which two or more values change linearly at the same time. Often you want to find when those values will be equal. For example, the values might be the location of two walkers, and you want to determine when the walkers meet.

Each growing value is represented by a linear equation. So several values changing linearly are represented by several linear equations, called a system of linear equations. Identifying where the values are equal requires solving the system, that is, finding values of the variables that make all the linear equations in the system true.

Discovering Algebra includes four methods for solving systems of two linear equations: by graphing, by substitution, by elimination, and by matrices.

Linear Inequalities
Another way to build conceptually on the idea of a linear equation is to change the equal sign to a less than or greater than sign. If you do that, you are stating that the two expressions are not equivalent.

This chapter introduces linear inequalities in which one of the variables has a known value, such as $5 + 2a > 21$. Such an inequality has infinitely many solutions, and you can visualize them on a number line. Students find the solutions by using the same methods they used for solving a linear equation in Chapter 3.

Then this chapter considers a linear inequality in two variables. Just as the pairs of numbers satisfying an equation can be represented by a line on a graph, the pairs of numbers satisfying an inequality can be represented on a graph. They appear as all points on one side of the line representing the corresponding equation. For a strict inequality, such as $y < 2x + 1$, points on the boundary line $y = 2x + 1$ make the inequality false, so the line is dashed to show that only the shaded portion of the graph, and not the boundary line, represents the solution.

Systems of Linear Inequalities
The mathematical methods called linear programming apply to many real-world situations. These methods rely on systems of linear inequalities. You can visualize the solutions to these systems graphically as the region containing only points that satisfy all inequalities in the system.

(continued)
Chapter 5 • Systems of Equations and Inequalities (continued)

Summary Problem
This summary problem is based on Lesson 5.4, Exercise 10. Will is baking bread. He has two different kinds of flour. Flour X is enriched with 0.12 mg of calcium per gram; Flour Y is enriched with 0.04 mg of calcium per gram. Each loaf has 300 g of flour, and Will wants each loaf to have 30 mg of calcium. How much of each type of flour should he use for each loaf?

If your student’s teacher covered Lesson 5.7, add the following modification: Will has started selling his bread and is having trouble making a profit. To keep his overall costs down, each loaf can contain at most 300 g of flour. He would like at least 25 g of calcium per loaf.

Discuss these questions and scenarios with your student in your role as student to your student:

- Write a system of equations to represent the problem and explain the meaning of each variable and each equation in the context of the problem.
- Which method for solving systems of equations would you choose to solve the system?
- Solve the system using one method. Then check by solving the system using a second method.
- Suppose Will is almost out of flour. He only has 275 g remaining. Can he still bake a loaf of bread with 30 mg of calcium?

Sample Answers
One possible system of equations is:

\[
\begin{align*}
  x + y &= 300 \\
  0.12x + 0.04y &= 30
\end{align*}
\]

The variable \(x\) represents the amount of Flour X in grams, and \(y\) represents the amount of Flour Y in grams. The first equation represents the restriction that each loaf has 300 g of flour. 0.12\(x\) represents the amount of calcium contributed by Flour X, and 0.04\(y\) represents the amount of calcium contributed by Flour Y. The second equation represents the total amount of calcium in the loaf.

Substitution, elimination, and matrices are probably the best choices for solving this system. Students might also choose to solve by graphing on their calculator. You might ask your student to solve this by different methods as you are reviewing different lessons. They should find that the loaf of bread requires 225 g of Flour X and 75 g of Flour Y. If the problem is changed to use only 275 g of flour, the solution will be 237.5 g of Flour X and 37.5 g of Flour Y.

If Will can use no more than 300 g of flour, and each loaf must have at least 25 g of calcium, the system of inequalities can represent the new situation:

\[
\begin{align*}
  x + y &\leq 300 \\
  0.12x + 0.04y &\geq 25
\end{align*}
\]

Students should test points to determine how to shade each inequality, and then find several points that satisfy both inequalities. Some sample solutions are (180, 100), representing 180 g of Flour X and 100 g of Flour Y, (215, 16), and (290, 0).
Chapter 5 • Review Exercises

1. (Lesson 5.1) Determine whether the ordered pair is a solution to the system of equations. Graph both lines in the system, and plot the point.
   a. \((1, -2)\) \(\begin{align*}
   y &= 2x - 4 \\
   y &= -x - 1
   \end{align*}\)
   b. \((3, 1)\) \(\begin{align*}
   y &= -\frac{2}{3}x + 3 \\
   y &= \frac{2}{3}x - 2
   \end{align*}\)

2. (Lesson 5.2) Solve this system of equations using the substitution method, and then check your answer.
   \(\begin{align*}
   2x + 3y &= 7 \\
   x - 4y &= -2
   \end{align*}\)

3. (Lesson 5.3) Solve this system by elimination, and then check your work.
   \(\begin{align*}
   2x - 3y &= -2 \\
   5x + 2y &= -5
   \end{align*}\)

4. (Lessons 5.1–5.3) Jenna purchased peaches and pears at the local market. The peaches cost $2.90 per pound, and the pears cost $1.10 per pound. Jenna bought a total of 8 pounds of fruit, which cost $18.34. How many pounds of each fruit did Jenna buy?

5. (Lesson 5.5) Solve the inequality \(3 - 5x \geq 8\) and graph the solutions on a number line.

6. (Lessons 5.6, 5.7) Consider the system of inequalities \(\begin{align*}
   y &< 2x - 3 \\
   y &\geq -x + 1
   \end{align*}\)
   a. Determine whether each of the following ordered pairs is a solution to this system of inequalities.
      i. \((0, 2)\)     ii. \((4, 2)\)     iii. \((-3, 1)\)     iv. \((3, -2)\)
   b. Graph the system of inequalities, and plot each of the points from 6a.
1. a. Yes. The ordered pair (1, -2) satisfies both equations.

\[ y = 2x - 4 \quad y = -x - 1 \]
\[ -2 = 2(1) - 4 \quad -2 = -1 - 1 \]
\[ -2 = -2 \]

b. No. The ordered pair (3, 1) does not satisfy the second equation.

\[ y = -\frac{2}{3}x + 3 \quad y = \frac{2}{3}x - 2 \]
\[ 1 \neq -\frac{2}{3}(3) + 3 \quad 1 \neq \frac{2}{3}(3) - 2 \]
\[ 1 \neq -2 + 3 \quad 1 \neq 2 - 2 \]
\[ 1 \neq 1 \]

2. (2, 1). Solve the second equation for \( x \), and substitute the expression into the first equation.

\[ x - 4y = -2 \quad \text{Second equation.} \]
\[ x = 4y - 2 \quad \text{Add 4y to both sides.} \]
\[ 2(4y - 2) + 3y = 7 \quad \text{Substitute 4y - 2 for } x \text{ in the first equation.} \]
\[ 8y - 4 + 3y = 7 \quad \text{Use the distributive property.} \]
\[ 11y - 4 = 7 \quad \text{Combine like terms.} \]
\[ 11y = 11 \quad \text{Add 4 to both sides.} \]
\[ y = 1 \quad \text{Divide both sides by 11.} \]

Substitute 1 for \( y \) into one of the original equations and solve for \( x \): \( x - 4(1) = -2; x = 2 \). The solution is (2, 1).

Check your solution.

\[ 2x + 3y = 7 \]
\[ 2(2) + 3(1) = 7 \]
\[ 7 = 7 \]

3. \((-1, 0)\)

\[ 2(2x - 3y) = 2(-2) \rightarrow 4x - 6y = -4 \quad \text{Multiply both sides by 2.} \]
\[ 3(5x + 2y) = 3(-5) \rightarrow 15x + 6y = -15 \quad \text{Multiply both sides by 3.} \]
\[ 19x = -19 \quad \text{Add the equations.} \]
\[ x = -1 \quad \text{Divide both sides by 19.} \]

Substitute this \( x \)-value into either of the original equations and solve to find \( y \). Using the first equation, \( 2(-1) - 3y = -2; y = 0 \).

Check your solution in both equations.

4. 5.3 lb of peaches, 2.7 lb of pears. Let \( x \) be the number of pounds of peaches, and \( y \) be the number of pounds of pears Jenna bought. Adding up pounds yields the equation \( x + y = 8 \), and adding up cost yields the equation \( 1.10x + 2.90y = 18.34 \). This system may be more easily solved by substitution.

5. \(3 - 5x \geq 8\)  \ Original inequality.
\[-5x \geq 5\quad \text{Subtract 3 from both sides.} \]
\[x \leq -1\quad \text{Divide both sides by } -5 \text{ and reverse the inequality.} \]

6. a. Substitute the values for \( x \) and \( y \) into each inequality and see whether a true statement results.

i. No. The ordered pair satisfies the second inequality but not the first.

ii. Yes. The ordered pair satisfies both inequalities.

iii. No. The ordered pair does not satisfy either inequality.

iv. Yes. The ordered pair satisfies both inequalities.

b. Begin by graphing the equations \( y = 2x - 3 \) and \( y = 1 - x \). Because the first inequality has \( < \) instead of \( \leq \), the line \( y = 2x - 3 \) should be dashed to indicate that it is not included in the solution. The other line should be solid. Plot the points and shade the area containing the solutions.
CHAPTER 6

Exponents and Exponential Models

Content Summary

Students understand a concept not only by seeing examples of the concept but also by seeing non-examples or counter-examples of that concept. For instance, if you try to teach a baby the concept of the color blue by pointing only to a variety of blue objects and saying “blue,” the child might think that “blue” describes everything. You would need to point to some non-blue objects and say their colors as well.

Similarly, there are many kinds of growth that are not linear. Students’ understanding of linear growth will deepen as they study other kinds of growth. They begin a study of nonlinear growth in Chapter 6, where they learn about exponential growth. After comparing linear and exponential growth in Chapter 7, they’ll encounter other types of nonlinear growth in Chapters 8 and 9. Even those people who consider the heart of elementary algebra to be linear equations acknowledge the importance of these other topics.

Exponential Growth

To find values that are growing linearly, you add the same amount repeatedly. In contrast, to find many real-world values, you multiply by the same amount repeatedly. For example, to find the cost of bread over several years, you might multiply by 1.02 for each year to account for an inflation rate of 2%. Or, to find the height of a ball from one bounce to the next, you might multiply by 0.85 to account for the loss of height from bounce to bounce. This kind of growth is called exponential growth (or decay).

To find the height of the ball after 9 bounces, given a starting height of 2 meters, you can multiply 2 by 0.85 repeatedly.

\[(2)(0.85)(0.85)(0.85)(0.85)(0.85)(0.85)(0.85)(0.85) = 0.4632\]

This method uses recursion, which students first encountered in Chapter 0. In contrast, if you use the shortcut to find the value by substituting 9 for x in \[y = 2(0.85)^x\], you’re using an exponential equation. The variable x is the number of times you use 0.85 as a factor.

Exponents

Working with exponential equations requires understanding the rules of exponents. If the ball bounces two more times after the nine bounces, you can find its height by evaluating either \(2(0.85)^1\) or \(2(0.85)^9(0.85)^2\). The fact that these two values are the same illustrates the multiplication property of exponents: \((0.85)^m \cdot (0.85)^n = (0.85)^{m+n}\).

More generally, if \(b\) is any base, \(b^m \cdot b^n = b^{m+n}\).

(continued)
Chapter 6 • Exponents and Exponential Models (continued)

You can also find the height of the ball three bounces before the ninth bounce in two ways. You could simply evaluate $2(0.85)^6$, or you could evaluate $\frac{2(0.85)^9}{(0.85)^3}$. The corresponding division property of exponents is $\frac{b^m}{b^n} = b^{m-n}$.

If $m = n$, on the left side of the preceding equation, you’re dividing a number by itself to get 1. On the right side, you get $b^0$. That’s why any number (except 0) to the 0 power is defined to be 1. If $m < n$, the exponent on the right is a negative number, and on the left, after cancellation, a positive power is in the denominator. These ideas are explored in the More Exponents Investigation in Lesson 6.6.

These properties of exponents are the reason that scientific notation is so useful when working with very large and very small numbers.

Summary Problem

You and your student might discuss this problem, expanded from Lesson 6.2, Exercise 13.

Your friend just bought an antique car for $5000. How much might it be worth in 10 years?

Questions you might ask, in your role as student to your student, include:

- What will be the car’s value for various rates at which that value might appreciate (increase exponentially)?
- What will be the car’s value for various rates at which that value might depreciate (decrease exponentially)?
- To find the car’s value in 12 years at one of those growth rates, can you use the multiplication property of exponents?
- To find the car’s value in 20 years at one of those growth rates, can you use a power property of exponents?
- To find the car’s value in 8 years at one of those growth rates, can you use the division property of exponents?

Sample Answers

Assuming a 6% increase, the equation $y = 5000(1 + 0.06)^x$ shows a value in 10 years of $y = 5000(1 + 0.06)^{10}$, or $8954$. You might take this answer and multiply it by the growth rate $(1 + 0.06)^2$ to get the value after 12 years, using the multiplication property of exponents. Or you can divide it by $(1 + 0.06)^2$ to get the value after 8 years, using the division property of exponents. The power property of exponents would indicate that the multiplier for 20 years is the square of the multiplier for 10 years, so $5000[(1 + 0.06)^{10}]^2 = 5000(1 + 0.06)^{20}$.

If instead the car depreciates at 1% per year for 10 years, its worth $y$ would be $5000(1 - 0.01)^{10}$, or about $4522.$
Chapter 6 • Review Exercises

1. (Lessons 6.1, 6.2) Yossi just started a new job. His beginning salary is $34,500, and he is guaranteed a 5% raise every year that his work remains satisfactory.
   a. Determine the multiplier used to calculate Yossi’s salary each year.
   b. Write a recursive routine and use it to find Yossi’s salary after five years.
   c. When will Yossi be making more than $50,000 per year?
   d. Write an exponential equation to model Yossi’s salary.

2. (Lesson 6.2) For the table below, find the values of the constants $a$ and $b$ such that $y = a \cdot b^x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.5</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
</tr>
<tr>
<td>5</td>
<td>0.02</td>
</tr>
<tr>
<td>8</td>
<td>0.00016</td>
</tr>
</tbody>
</table>

3. (Lessons 6.3, 6.5) Use the properties of exponents to rewrite each expression.
   a. $2(3x^3)^2(x^5)$
   b. $\frac{(3x^2)^4}{27x^3}$
   c. $(-2x^4y^2)(3x^5y)$

4. (Lessons 6.4, 6.6) Rewrite each number in scientific notation.
   a. 12,300,000
   b. 0.00004
   c. $314 \times 10^4$

5. (Lessons 6.6, 6.7) Sal monitored the amount he had of a certain radioactive isotope every hour, and recorded the data in the table shown at right.
   a. Estimate the percentage decrease in the amount of the isotope per hour.
   b. Write an exponential equation to model the amount of isotope. Graph the equation along with a scatter plot of the data.
   c. Use your model from 5b to estimate the amount of isotope 7 hours prior to Sal beginning to record his data.

<table>
<thead>
<tr>
<th>Time elapsed (h)</th>
<th>Amount (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>68.3</td>
</tr>
<tr>
<td>1</td>
<td>57.7</td>
</tr>
<tr>
<td>2</td>
<td>52.1</td>
</tr>
<tr>
<td>3</td>
<td>43.2</td>
</tr>
<tr>
<td>4</td>
<td>38.1</td>
</tr>
<tr>
<td>5</td>
<td>31.4</td>
</tr>
<tr>
<td>6</td>
<td>26.9</td>
</tr>
</tbody>
</table>
**SOLUTIONS TO CHAPTER 6 REVIEW EXERCISES**

1. a. \((1 + 0.05)\), or 1.05.
   b. \([0, 34500]\)  
   Ans(1) + 1, Ans(2) \cdot (1 + 0.05)  
   After five years, Yossi's salary will be $44,031.71.

c. After eight years, his salary will exceed $50,000.

d. \(y = 34,500(1 + 0.05)^x\), where \(y\) is Yossi's salary after \(x\) years. In the exponential equation \(y = A(1 + r)^x\), the value \(r\) is the rate of growth; and \(A\) is the initial value, or the \(y\)-value when \(x = 0\).

2. \(y = 62.5 \cdot 0.2^x\). Use the \(y\)-value corresponding to consecutive \(x\)-values to find the multiplier: 
   \(\frac{2.5}{12.5} = 0.2\) and \(\frac{0.2}{0.02} = 0.2\), so \(b = 0.2\). The value of \(a\) is the \(y\)-value when \(x = 0\). To find this, divide 12.5 by the multiplier: \(\frac{12.5}{0.2} = 62.5\), so \(a = 62.5\).

3. a. \(2(3x^3)^2(x^5)^3\)  
   Original expression.  
   \(2 \cdot 3^2x^{3\cdot2} \cdot x^5\)  
   Use the power properties of exponents.  
   \(2 \cdot 9x^6 \cdot x^5\)  
   Multiply.  
   \(18x^{6+5}\)  
   Use the multiplication property of exponents.  
   \(18x^{11}\)  
   Add.  
   b. \(\frac{(3x^3)^4}{27x^3}\)  
   Original expression.  
   \(\frac{3^4x^{3\cdot4}}{3^3x^3}\)  
   Use the power properties of exponents.  
   \(3x^{8-3}\)  
   Use the division property of exponents.  
   \(3x^5\)  
   Subtract.

c. \((-2x^4y^2)^3(3x^3y)\)  
   Original expression.  
   \((-8x^{12}y^6)(3x^3y)\)  
   Use the power properties of exponents.  
   \(-24x^{15}y^7\)  
   Multiply coefficients and use the multiplication property of exponents.

4. a. \(12,300,000 = 1.23 \times 10^7\)
   b. \(0.00004 = 4.0 \times 10^{-5}\)
   c. \(314 \times 10^4 = 3.14 \times 10^2 \times 10^4 = 3.14 \times 10^6\)

5. a. Find the ratios from each \(y\)-value to the next, and find their mean. The mean of the ratios is approximately 0.86, which is \(1 - 0.14\). This means the amount of the substance decreased by approximately 14% each hour.
   b. \(y = 68.3(1 - 0.14)^x\), where \(y\) is the number of grams of the isotope \(x\) hours after Sal began recording the data.

   \([0, 10, 1, 0, 75, 10]\)

c. Use the model from 5b with \(x = -7\).
   \(y = 68.3(1 - 0.14)^{-7} \approx 196.3\). There were approximately 196.3 g of the isotope 7 hours before Sal began to record his data.
CHAPTER 7

Functions

Content Summary

In Chapter 7, students increase their understanding of linear growth and equations by looking in detail at the special kind of relation called a function. This section helps students to become aware of functions by creating, reading, and describing codes. They are also introduced to other kinds of nonlinear growth: absolute values, squares, and square roots.

Functions and Graphs

In ordinary conversation, we might say something like “Her pay is a function of how long she’s been in the job.” In this sense, the phrase is a function of would mean is dependent upon. In mathematics, the dependence is taken literally; if pay were a function of time in the job, then any two people who had spent the same time in the same job would earn exactly the same pay.

<table>
<thead>
<tr>
<th>Employee</th>
<th>Time in the job</th>
<th>Pay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jack</td>
<td>6 months</td>
<td>$8.50</td>
</tr>
<tr>
<td>Jason</td>
<td>12 months</td>
<td>$9.00</td>
</tr>
<tr>
<td>Jim</td>
<td>18 months</td>
<td>$9.50</td>
</tr>
<tr>
<td>Julie</td>
<td>24 months</td>
<td>$10.00</td>
</tr>
<tr>
<td>June</td>
<td>12 months</td>
<td>$9.00</td>
</tr>
</tbody>
</table>

If you consider the last two columns in the preceding table as a data table, no two pairs of numbers that have the same first number will have different second numbers. When you translate those numbers into points with the number pairs as coordinates, you find that no two data points lie on the same vertical line. That is, no vertical line will cross the graph of a function more than once.

Equations in which one variable is expressed in terms of a second variable can define a function. Instead of writing \( y = 3 + 5x \), you might write \( f(x) = 3 + 5x \). The left side, read “f of x,” is the name you give to the function with x as the independent variable. If you want to find the value of the function when \( x = 7 \), you write \( f(7) = 3 + 5(7) \), or \( f(7) = 38 \). The graph of this function would include the point with coordinates (7, 38).

Absolute-Value Functions

The absolute-value function extends students’ understanding of linearity. The absolute value of a number is, informally, its “positive value.” If the number is positive or 0, then its absolute value is itself. If a number is negative, then its absolute value is its opposite, the corresponding positive number. For example, the absolute value of 3, written \( |3| \), is 3 itself, because 3 is positive. The absolute value of \(-3\), however, written \( |-3| \), is 3.

(continued)
Chapter 7 • Functions (continued)

The graph of the absolute-value function \( f(x) = |x| \) consists of two rays that make
the shape of a letter V with its bottom point at the origin. Realizing why the
absolute-value function has this graph helps students understand both function
graphs and absolute values.

Squaring and Square Root Functions

The squaring and square root functions are also important nonlinear functions
providing a contrast to linear functions. The square function simply squares a
number, multiplying it by itself. It's a function because no number has two
different squares. Its graph is a parabola opening upward with its lowest point
(vertex) at the origin.

But interchanging the two columns of the squaring data table does not yield a
function. For example, the number 4 in the first column would appear once with
2 in the second column and once with \(-2\). That’s because \(2^2\) and \((-2)^2\) are both 4.
In other words, 4 has two square roots: 2 and \(-2\). In this sense, taking the square
root is not a function.

An associated function, however, often called the square root function, produces the
positive square root. The standard square root symbol \(\sqrt{}\) means the positive
square root. The graph of the square root function is the top half of a parabola that
opens to the right.
Chapter 7 • Functions (continued)

Summary Problem

Make a code this way: Change each letter of the alphabet to an integer between 1 and 26. Square the number, and subtract 26 repeatedly until you get a number between 1 and 26. Change that number back to its corresponding letter. How does the assignment of each letter to another letter relate to the ideas of this chapter?

Questions you might ask in your role as student to your student include:

- Does the code assignment create a function?
- How does decoding work?
- Is there a sense in which some integers from 1 to 26 are the opposites of other integers from 1 to 26?
- Could the ideas of positive and absolute value be interpreted for the integers 1 to 26 so that the positive square root of a number is the absolute value of either of its square roots?
- Can you do similar arithmetic with integers from 1 to some integer other than 26?

Sample Answers

You and your student might start with a table that shows what each letter becomes in the code.

<table>
<thead>
<tr>
<th>Letter</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
<th>L</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Code</td>
<td>A</td>
<td>D</td>
<td>I</td>
<td>P</td>
<td>Y</td>
<td>J</td>
<td>W</td>
<td>L</td>
<td>C</td>
<td>V</td>
<td>Q</td>
<td>N</td>
<td>M</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Letter</th>
<th>N</th>
<th>O</th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
<th>T</th>
<th>U</th>
<th>V</th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Code</td>
<td>N</td>
<td>Q</td>
<td>V</td>
<td>C</td>
<td>L</td>
<td>W</td>
<td>J</td>
<td>Y</td>
<td>P</td>
<td>I</td>
<td>D</td>
<td>A</td>
<td>Z</td>
</tr>
</tbody>
</table>

As you look at the table and think about how decoding would work, you see that for many letters in the coded message, there are two ways to decode the word. The coded word NAWJ might be NAGE, NAST, NAGT, LAST, and so on. Only one of these is a word, so it might be possible to decode a message, but it would be a complicated task.

Your student may recognize that the second half of the alphabet is a reflection of the first half, just as positive and negative integers are reflections of each other across zero. If it were agreed that the sender would use only the first half or only the last half of the alphabet, then the decoding would be unique. This would be like using only positive square roots.

The same phenomenon holds with integers from 1 up to any number.
Chapter 7 • Review Exercises

1. (Lesson 7.1) Use the given coding grid to answer 1a–c.
   a. Code the word ALGEBRA.
   b. Decode the word KSMHSN.
   c. Is this code a function? Explain why or why not.

2. (Lesson 7.2) Determine whether each of the following relationships represents a function. For each relationship that does represent a function, state the domain and range.

<table>
<thead>
<tr>
<th>Input $x$</th>
<th>Output $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

3. (Lessons 7.2–7.4) Ted drove straight to a paint supply store, bought some paint, and then drove straight home. The graph of the function $y = f(x)$ shows Ted’s distance from his house as a function of time.
   a. What is $f(20)$? What is the real-world meaning of $f(20)$?
   b. How far is the paint supply store from Ted’s house?
   c. How long was Ted at the paint supply store?
   d. On Ted’s way to the store, all the traffic lights were green. How many red traffic lights did he encounter on his way home? How long was Ted stopped at each red light?

4. (Lessons 7.4, 7.6) Consider the function $f(x) = |x + 2| - 1$.
   a. Graph the function.
   b. Find the values $f(1)$ and $f(-4)$. Which points do these values give you on the graph of $f(x)$?
   c. Find all $x$-values for which $f(x) = 1$. 
1. a. Find each letter on the bottom of the grid, and go up the column to a shaded square. Then go to the left of the square to find the coded letter. ALGEBRA codes into GLYSJHG.

b. Find each coded letter on the left of the grid, and go across the row to a shaded square. Then go down from the square to find the input letter. KSMHSN decodes into SECRET.

c. Yes, it is a function. Each input letter is assigned to only one coded letter. That is, there is only one shaded square in each column.

2. The domain is the set of all possible input values, or \( x \)-values; and the range is the set of all possible output values, or \( y \)-values.

   a. Not a function. The \( x \)-value 2 corresponds to two different \( y \)-values, 10 and 7.

   b. This is a function. Each \( x \)-value corresponds to only one \( y \)-value. Domain: \( \{1, 2, 3, 5, 7, 8\} \); Range: \( \{1, 2, 3, 4\} \).

   c. This is a function. Each \( x \)-value corresponds to only one \( y \)-value. Any vertical line crosses the graph of the function only once. Domain: \( 0 \leq x \leq 4 \); Range: \( -1 \leq y \leq 3 \).

   d. Not a function. The value \( x = 0 \) corresponds to more than one \( y \)-value. A vertical line through 0 crosses the graph of the function more than once.

3. a. See what point on the graph has 20 as its first coordinate. The point (20, 2) is on the graph, so \( f(20) = 2 \).

   b. Notice from the graph that the farthest Ted got away from his house was 4 mi. Therefore, the store is 4 mi from Ted’s house.

   c. Ted was at the store from the 8 min mark until the 15 min mark, so he was there for 7 min.

   d. The parts of the graph that are level indicate that Ted is not moving. There are three of these on his way home, and they last for one minute each.

4. a. 

   b. \[ f(1) = \left| 1 + 2 \right| - 1 \] Substitute 1 for \( x \).

   \[ f(1) = \left| 3 \right| - 1 \] Add.

   \[ f(1) = 3 - 1 \] Take the absolute value.

   \[ f(1) = 2 \] Subtract.

   \[ f(-4) = \left| -4 + 2 \right| - 1 \] Substitute \(-4\) for \( x \).

   \[ f(-4) = \left| -2 \right| - 1 \] Add.

   \[ f(-4) = 2 - 1 \] Take the absolute value.

   \[ f(-4) = 1 \] Subtract.

   The points (1, 2) and \((-4, 1)\) are on the graph of the function.

c. \( x = -4 \) and \( x = 0 \). You can solve this graphically by finding the intersection of the graph of \( f(x) \) with the horizontal line \( y = 1 \). The symbolic solution is shown below.

\[ \left| x + 2 \right| - 1 = 1 \] Equation to be solved.

\[ \left| x + 2 \right| - 1 + 1 = 1 + 1 \] Add 1 to both sides.

\[ \left| x + 2 \right| = 2 \] Add.

\[ x + 2 = 2 \] or \( x + 2 = -2 \) Find two numbers whose absolute value is 2.

\[ x + 2 - 2 = 2 - 2 \] Subtract 2 from both sides of each equation.

\[ x = 0 \] or \( x = -4 \) Subtract.
CHAPTER 8

Transformations

Content Summary

In Chapter 8, students continue their work with functions, especially nonlinear functions, through further study of function graphs. In particular, they consider three ways of changing the location, orientation, and size of those graphs. (Note: You might skip the material on matrices if your student’s teacher is not covering Lesson 8.7; matrices are often covered as part of an advanced algebra curriculum.)

Translations

A translation shifts points or graphs on the plane. If a point \((x, y)\) is shifted to the right \(h\) units and up \(k\) units, the resulting point is \((x + h, y + k)\).

A function graph can be shifted in the same way, by replacing \(x\) with \(x + h\) in each occurrence of \(x\) in the graph’s equation, and by replacing \(y\) with \(y + k\) in each occurrence of \(y\).

As an example, consider the rational function \(f(x) = \frac{1}{x}\), which the book introduces in this chapter. If you shift the graph of \(y = \frac{1}{x}\) to the left 2 units and up 3 units, the result has the equation \(y + 3 = \frac{1}{x + 2}\). You can also think of the resulting equation as the graph of function \(g(x) = \frac{1}{x + 2} + 3\). Although \(f(x) = \frac{1}{x}\) is undefined for \(x = 0\) and has \(y = 0\) as an asymptote (a line that the graph approaches but never touches), \(g(x)\) is undefined 2 units to the left, where \(x = -2\), and has an asymptote 3 units higher, at \(y = 3\).

It is possible to represent transformations with matrices. In particular, a translation can be represented by matrix addition. To do so, you can represent the point \((x, y)\) by the \(2 \times 1\) matrix \[
\begin{bmatrix}
x \\
y
\end{bmatrix}
\] and the translation by \[
\begin{bmatrix}
h \\
k
\end{bmatrix}
\]. Then the image point is represented by the sum of these matrices: \[
\begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
h \\
k
\end{bmatrix} = \begin{bmatrix}
x + h \\
y + k
\end{bmatrix}.
\]

In fact, this matrix approach can represent the translation of more than one point. *Discovering Algebra* shows how a matrix can represent the vertices (corners) of a polygon, with each column being the coordinates of a vertex. For example, the matrix \[
\begin{bmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4
\end{bmatrix}
\] represents the pentagon pictured on the next page. A shift to the right 3 units and up 2 units can be represented by the matrix \[
\begin{bmatrix}
3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2
\end{bmatrix}.
\]
Chapter 8 • Transformations (continued)

The resulting polygon is represented by the sum of the matrices:
\[
\begin{bmatrix} 3 & 1 & -2 & -1 & 2 \\ 2 & -1 & 0 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 1 & 2 & 5 \\ 4 & 1 & 2 & 4 & 6 \end{bmatrix}
\]

It is shown as the dashed polygon on the graph.

Reflections

The book examines reflections (or flips) of points and graphs across the axes.

When you reflect a point \((x, y)\) across the \(y\)-axis, the result is the point \((-x, y)\). Reflecting the point \((x, y)\) across the \(x\)-axis yields an image of \((x, -y)\).

To reflect a graph across the \(y\)-axis, you replace each occurrence of \(x\) in its equation with \(-x\). To reflect a graph across the \(x\)-axis, you replace \(y\) with \(-y\). For example, the reflection of the graph of \(y = \frac{1}{x+2}\) across the \(y\)-axis has equation \(f(x) = -\frac{1}{x+2}\). The reflection of \(y = \frac{1}{x+2}\) across the \(x\)-axis has equation \(-y = \frac{1}{x+2}\), or \(y = -\frac{1}{x+2}\).

Reflections can also be represented by matrix multiplication. For example, to reflect the pentagon represented by:
\[
\begin{bmatrix} 3 & 1 & -2 & -1 & 2 \\ 2 & -1 & 0 & 2 & 4 \end{bmatrix}
\]
across the \(y\)-axis, multiply by the matrix:
\[
\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 \end{bmatrix}
\]
to get:
\[
\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & -1 & 2 \\ 2 & -1 & 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 2 & 1 & -2 \\ 2 & -1 & 0 & 2 & 4 \end{bmatrix}
\]

Stretches and Shrinks

A vertical stretch by positive factor \(a\) changes \((x, y)\) to \((x, ay)\). If \(a\) is less than 1 (but still positive), the stretch is a shrink.

To stretch or shrink a graph vertically by positive factor \(a\), replace each occurrence of \(y\) with \(\frac{y}{a}\) in the equation of the graph. For example, replacing \(y\) with \(\frac{y}{2}\) in the equation \(f(x) = \frac{1}{x}\) creates a vertical stretch by a factor of 2. This is the equation of the function \(g(x) = \frac{2}{x}\). (continued)
Stretches and shrinks can also be represented by matrix multiplication. You can find the image of pentagon \[
\begin{pmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4 \\
\end{pmatrix}
\] after it is stretched vertically by a factor of \(\frac{1}{3}\) if you multiply by the matrix \[
\begin{pmatrix}
1 & 0 \\
0 & \frac{1}{3} \\
\end{pmatrix}
\] to get \[
\begin{pmatrix}
3 & 1 & -2 & -1 & 2 \\
2 & -1 & 0 & 2 & 4 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & \frac{1}{3} \\
\end{pmatrix}
= \begin{pmatrix}
\frac{2}{3} & 1 & -2 & -1 & 2 \\
\frac{2}{3} & -1 & 0 & \frac{2}{3} & 4 \\
\end{pmatrix}.
\]

### Summary Problem

<table>
<thead>
<tr>
<th>x</th>
<th>-1.5</th>
<th>-0.7</th>
<th>-0.2</th>
<th>0</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>-3.0</td>
<td>-1.2</td>
<td>-0.8</td>
<td>-0.7</td>
<td>-0.5</td>
<td>-0.3</td>
</tr>
</tbody>
</table>

Use transformations of the function \(f(x) = x\) to fit the data in the preceding table as well as you can.

Questions you might ask in your role as student to your student include:

- Why do different translations give the same result?
- Why do different reflections give the same result?
- Why do different combinations of stretches and shrinks give the same result?
- Does the order in which you do the transformations matter?
- Could a starting function other than \(f(x) = x\) be transformed to fit the data points better?

### Sample Answers

For a line with equation \(y = a + bx\), replacing \(x\) with \(x - h\) gives \(y = a + bx - bh\), or \(y + bh = a + bx\). So a horizontal translation by \(h\) is the same as a vertical translation by \(-bh\). Algebra also shows that a reflection of a line across the \(x\)-axis can be accomplished by a reflection across the \(y\)-axis, combined with a translation, and that stretching a line in one direction is equivalent to shrinking it in the perpendicular direction. A nonlinear function will probably fit the data better.
Chapter 8 • Review Exercises

1. (Lessons 8.1, 8.3, 8.4) Draw this triangle on graph paper or on your calculator. Then draw the image under each of the following transformations. Describe each transformation.
   a. \((x - 2, y + 1)\)  
   b. \((-x, y)\)  
   c. \((x, -y)\)  
   d. \((0.5x, 3y)\)

2. (Lessons 8.2–8.4) The graph of the function \(y = |x - 3|\) is shown below. Name the functions that give the following transformations of the graph. Check each answer by graphing it on your calculator.
   a. Translate right 2 units.
   b. Reflect across the \(y\)-axis and translate up 1 unit.
   c. Reflect across the \(x\)-axis, shrink vertically by a factor of 0.5, translate left 1 unit, and translate up 3 units.

3. (Lesson 8.6) Reduce each expression to lowest terms. State any restrictions on the variable.
   a. \(\frac{(2x^2)(10x^4)}{4x^3}\)  
   b. \(\frac{2x^2 - 4x}{6x}\)  
   c. \(\frac{4x(x - 3)}{2(x - 3)^2}\)  
   d. \(\frac{6 - 2x^2}{2x}\)
1. a. Translate left 2 units and up 1 unit.
   b. Reflect across the y-axis.
   c. Reflect across the x-axis.
   d. Shrink horizontally by a factor of 0.5; stretch vertically by a factor of 3.

2. a. \( y = |x - 3| \)
   Original function.
   \( y = |(x - 2) - 3| \)
   Replace \( x \) with \( x - 2 \) to translate the graph right 2 units.
   \( y = |x - 5| \)
   \([-9.4, 9.4, 1, -6.2, 6.2, 1]\)
   b. \( y = |x - 3| \)
   Original function.
   \( y = |-(x - 3)| \)
   Replace \( x \) with \(-x\) to reflect the graph across the y-axis.
   \( y - 1 = |-(x - 3)| \)
   Replace \( y \) with \( y - 1 \) to translate the graph up 1 unit.
   \( y = |-(x - 3)| + 1 \)
   \([-9.4, 9.4, 1, -6.2, 6.2, 1]\)
   c. \( y = |x - 3| \)
   Original function.
   \(-y = |x - 3| \)
   Replace \( y \) with \(-y\) to reflect across the x-axis.
   \( y = -|x - 3| \)
   Solve for \( y \).
   \( 2y = -|x - 3| \)
   Replace \( y \) with \( \frac{y}{2} \) or \( 2y \) to shrink the graph vertically by a factor of 0.5.

3. a. \( \frac{(2x^2)(10x^4)}{4x^3} = \frac{2 \cdot 2 \cdot 5 \cdot x^6}{2 \cdot 2 \cdot x^3} = 5x^3 \), where \( x \neq 0 \)
   The restriction \( x \neq 0 \) is necessary because the \( x \)-value 0 would make the denominator of the original expression zero.
   b. \( \frac{2x^2 - 4x}{6x} = \frac{2x(x - 2)}{2 \cdot 3 \cdot x} = \frac{x - 2}{3} \), where \( x \neq 0 \)
   The restriction \( x \neq 0 \) is necessary because the \( x \)-value 0 would make the denominator of the original expression zero.
   c. \( \frac{4x(x - 3)}{2(x - 3)^2} = \frac{2x}{x - 3} \), where \( x \neq 3 \).
   d. \( \frac{6 - 2x^2}{2x} = \frac{2(3 - x^2)}{2 \cdot x} = \frac{3 - x^2}{x} \), where \( x \neq 0 \).
Content Summary
In Chapter 9 the Discovering Algebra text continues to deepen students’ understanding of linear functions through the study of nonlinear functions. This chapter focuses on quadratic functions, eventually arriving at cubic functions.

Forms of Quadratic Equations
Quadratic equations can take three useful forms:

- The vertex form is \( y = a(x - h)^2 + k \). This form is useful for telling how the parent graph \( y = x^2 \) has been transformed. The vertex \((h, k)\) of the parabola is at the highest or lowest point. The factor \(a\) tells the amount of vertical stretch, and a negative value of \(a\) reveals a reflection across the \(x\)-axis.

- The factored form is \( y = a(x - x_1)(x - x_2) \). From this form it’s easy to tell that the roots of the equation are \(x_1\) and \(x_2\) and that the graph has \(x\)-intercepts at \(x_1\) and \(x_2\).

- The general form is \( y = ax^2 + bx + c \). This form is useful for finding the \(y\)-intercept to be \(c\)—the parabola crosses the \(y\)-axis at \((0, c)\). If the equation happens to be describing the height of a rising or falling object, then \(-a\) is half the acceleration due to gravity, \(b\) is the initial velocity, and \(c\) is the initial height above ground.

Here is an equation written in these three forms and its graph.

**Vertex form:** \( y = 2(x - 1)^2 - 8 \)

**Factored form:** \( y = 2(x - 3)(x + 1) \)

**General form:** \( y = 2x^2 - 4x - 6 \)

Changing Forms
Because the three forms serve different purposes, converting between them is common. The vertex and factored forms can be changed to the general form by multiplying binomials and combining like terms. The general form can be changed to the vertex form by completing the square. The general form can be changed to the factored form by factoring. Both multiplying and factoring can be aided by a rectangle diagram.

This rectangle diagram shows that \((x + 3)(x + 5) = x^2 + 8x + 15\).

\[
\begin{array}{|c|c|}
\hline
x & 3 \\
\hline
x^2 & 3x \\
\hline
5x & 15 \\
\hline
\end{array}
\]

You may have learned a procedure abbreviated F.O.I.L—First, Outside, Inside, Last—for multiplying binomials. Don’t push your student to use this method. The rectangle diagram provides a visual organizer to help students factor or multiply expressions. It is easily expanded to multiply binomials by trinomials. (Use a 2-by-3 rectangle.) The primary reason to change to the factored form is to solve the

(continued)
Chapter 9 • Quadratic Models (continued)

equation—to find its roots. Often factoring is very difficult or even impossible. A method of solving quadratic equations, regardless of whether the equations can be factored, is to use the quadratic formula, which is introduced in Lesson 9.7.

Be careful—students tend to confuse the terms quadratic equation and quadratic formula.

Cubic Functions

Cubic functions (with general form \( y = ax^3 + bx^2 + cx + d \)) frequently arise in real-world problems. To find the roots of these equations, Discovering Algebra uses graphs. Once you find the roots, you can derive the factored form of the cubic equation. Later, in Discovering Advanced Algebra, students will see how to factor cubic equations from the general form.

Summary Problem

You and your student might revisit this summary problem several times while working through the chapter.

The height of a particular model rocket is described by the quadratic function \( h(t) = \frac{1}{2}(-9.8)t^2 + 49t + 2.5 \), where \( t \) represents the number of seconds after the launch. What can you learn about the rocket’s height from this equation and other forms of this equation?

Questions you might ask in your role as student to your student include:

- What do the coefficients \(-9.8, 49, \text{ and } 2.5\) represent?
- What units are used to describe the rocket’s height?
- What is the vertex form of the equation?
- What can you learn from the vertex form?
- At what times is the height 0?
- When does the rocket reach ground level?
- How do the zeros of a quadratic or cubic function relate to its factored form?
- What is the factored form of the equation?
- What can you learn from the factored form?
- At what times is the rocket 20 meters above ground level?
- Are there other useful forms of quadratic equations?

Sample Answer

The coefficients represent the opposite of half the force due to gravity, the initial velocity, and the initial height in meters. Students can find the vertex form by completing the square to get \( h(t) = -4.9(t - 5)^2 + 125 \), from which they can tell that the rocket reaches its maximum height of 125 meters when \( t = 5 \). The zeros can best be found by using the quadratic formula to solve the equation \(-4.9t^2 + 49t + 2.5 = 0\). The zeros are at approximately \(-0.05\) and \(10.05\), so the factored form of the function is \( h(t) = -4.9(t + 0.05)(t - 10.05) \). The zeros show where the graph crosses the horizontal axis, or reaches ground level. The rocket hits the ground after 10.05 seconds. To find when the height is 20 meters, solve the equation \(20 = -4.9t^2 + 49t + 2.5\) for \( t \). (\( t \) is about 0.4 and 9.6.)
Chapter 9 • Review Exercises

Name ____________________________ Period ____________ Date ____________

1. (Lesson 9.1) Use a symbolic method to find the exact solutions of the equation \(2(x - 1)^2 + 5 = 9\). Express the solutions in radical form.

2. (Lesson 9.2) Name the vertex of the parabola given by each quadratic function. Then graph each equation to verify your answer.
   a. \(y = (x - 6)^2 - 3\)  
   b. \(y = 2(x + 1)^2 + 3\)

3. (Lesson 9.3) Convert the equation \(y = 2(x - 4)^2 - 7\) to general form. Check your answer by entering both equations into the Y= screen on your calculator and comparing their graphs.

4. (Lesson 9.4) Write the equation \(y = x^2 - 2x - 8\) in factored form, and then use the zero-product property to find the \(x\)-intercepts of the parabola described by the equation.

5. (Lesson 9.6) Complete the square to rewrite each equation in vertex form, and name the vertex of the parabola described by the equation.
   a. \(y = x^2 - 6x + 4\)  
   b. \(y = 3x^2 + 6x - 4\)

6. (Lesson 9.7) Use the quadratic formula to solve the equation \(2x^2 - 6x + 3 = 0\). Express the solutions in radical form.

7. (Lesson 9.8) Subtract the following rational expressions, and express your answer in reduced form. State any restrictions on the variable.
   \[
   \frac{3}{x - 1} - \frac{9x}{x^2 + x - 2}
   \]
3. First use a rectangle diagram to square the binomial \((x - 4)^2\).
\[ (x - 4)^2 = x^2 - 8x + 16 \]

\[ y = 2(x - 4)^2 - 7 \]
Original equation.

\[ y = 2(x^2 - 8x + 16) - 7 \]
Square the binomial.

\[ y = 2x^2 - 16x + 32 - 7 \]
Use the distributive property.

\[ y = 2x^2 - 16x + 25 \]
Combine like terms.

Graph both the vertex form, \(y = 2(x - 4)^2 - 7\), and the general form, \(y = 2x^2 - 16x + 25\), on your calculator to see that they produce the same graph.

4. Use a rectangle diagram to help you factor the polynomial.

Find two numbers whose product is \(-8\) and whose sum is \(-2\). These numbers are \(-4\) and \(2\), so the factored form of \(x^2 - 2x - 8\) is \((x - 4)(x + 2)\). To find the \(x\)-intercepts, solve the equation \((x - 4)(x + 2) = 0\).

\[(x - 4)(x + 2) = 0\] 
Equation.

\[ x - 4 = 0 \quad \text{or} \quad x + 2 = 0 \] 
Use the zero-product property.

\[ x = 4 \quad \text{or} \quad x = -2 \] 
Solve each equation.

The \(x\)-intercepts are 4 and \(-2\).

5. a. \(y = x^2 - 6x + 4\) 
Original equation.

Add \((-3)^2\), or 9, to make a perfect-square trinomial. You must also subtract 9 to keep the equation balanced.

\[ y = x^2 - 6x + 9 - 9 + 4 \]
Add zero in the form \(9 - 9\).

\[ y = (x - 3)^2 - 5 \]
Factor the trinomial.

The vertex is at \((3, -5)\).

b. \(y = 3x^2 + 6x - 4\)

\[ y = 3(x^2 + 2x + 1 - 1) - 4 \]
\[ y = 3(x^2 + 2x + 1) + 3(-1) - 4 \]
\[ y = 3(x + 1)^2 - 3 \]
\[ y = 3(x + 1)^2 - 7 \]
The vertex is at \((-1, -7)\).

6. The solutions to the quadratic equation \(ax^2 + bx + c = 0\) are \(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\). For the given equation \(2x^2 - 6x + 3 = 0\), \(a = 2\), \(b = -6\), and \(c = 3\). Therefore the solutions are \(x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(3)}}{2(2)}\), or \(x = \frac{6 \pm \sqrt{12}}{4}\).

7. Factor \(x^2 + x - 2\) to find a common denominator:
\[ x^2 + x - 2 = (x + 2)(x - 1) \]

\[ \frac{3}{x - 1} - \frac{9x}{x^2 + x - 2} \]
Original expression.

\[ = \frac{3}{x - 1} \cdot \frac{x + 2}{x + 2} - \frac{9x}{(x - 1)(x + 2)} \]
Multiply by 1 to get a common denominator.

\[ = \frac{3(x + 2)}{(x - 1)(x + 2)} - \frac{9x}{(x - 1)(x + 2)} \]
Multiply.

\[ = \frac{3x + 6 - 9x}{(x - 1)(x + 2)} \]
Use the distributive property.

\[ = \frac{3x + 6 - 9x}{(x - 1)(x + 2)} \]
Add the numerators.

\[ = \frac{-6x + 6}{(x - 1)(x + 2)} \]
Combine like terms.

\[ = \frac{-6(x - 1)}{(x - 1)(x + 2)} \]
Factor the numerator.

\[ = -\frac{6}{x + 2} \]
where \(x \neq 1\) and \(x \neq -2 \)
Reduce the expression.

The restriction \(x \neq 1\) is necessary because the \(x\)-value 1 makes the denominators of the original expression zero.
CHAPTER 10

Probability

Content Summary

Chapter 10 introduces basic probability concepts, including special kinds of events, expected value, and counting permutations and combinations.

Relative Frequency Graphs and Probability

Chapter 10 introduces relative frequency graphs, which display categorical data. Relative frequency bar and circle graphs display the percent or fraction of each category relative to the total for all categories.

The chance of something occurring, or the probability of an outcome, can be determined from a relative frequency graph. For example, for a randomly chosen item from the library collection, the probability that the item is adult fiction is 24%, or 0.24. An experimental probability, or observed probability, is based on data or experiments and is defined as \( \frac{\text{number of occurrences of event}}{\text{total number of trials}} \). A theoretical probability, defined as \( \frac{\text{number of different ways the event can occur}}{\text{total number of equally likely outcomes possible}} \), uses known quantities. For a fair coin, the theoretical probability of getting a head is 50%, because heads and tails are equally likely. However, in flipping a coin, someone might get a run of heads or tails that leads them to a different experimental probability for heads. After many coin tosses, the experimental probability for heads would approach 50%.
Chapter 10 • Probability (continued)

Independent Events
If you flip a coin repeatedly and get 5 heads in a row, you might say that the chance, or probability, of getting a head on the next flip is very small. After all, the chance of getting 6 heads in a row is very small. Or, you might think that the chance of getting a head on the next flip is large; that there’s a “run” on heads. In fact, however, the coin has no memory; the chance of getting a head on the next flip is 0.5, as it has been all along. You can say that the events are independent; the result of the sixth flip does not depend on the result of the fifth flip.

In the case of independent events, the probability that both will occur is the product of the probabilities of the individual events. The probability of getting heads five times in a row is \( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32} \). This is also the probability of getting any particular string of heads or tails. In other words, the probability of getting T T H T T is also \( \frac{1}{32} \).

Not all events are independent of previous events. Suppose you have a bag with six bills, five 1-dollar bills and one 100-dollar bill. The chance of selecting the 100-dollar bill is 1 in 6, or about 0.17. However, if someone selects a one-dollar bill and removes it from the bag, the next person has a 1 in 5, or 0.2 probability, of picking the 100-dollar bill. Of course, if the first person picks the 100-dollar bill, then the next person has no chance, or 0 probability, of choosing the 100-dollar bill.

Permutations and Combinations
Determining the numbers to calculate theoretical probabilities can be challenging.

Sometimes the outcomes to be counted are arrangements of things or people. For example, suppose ten people attend a meeting, and you randomly pick three of them to win different door prizes. Any of the ten could get door prize A, the most valuable. Any of the remaining nine could win door prize B, the next most valuable. And any of the remaining eight could win the third door prize, C. There are \( 10 \cdot 9 \cdot 8 = 720 \) ways that three of the ten people could be arranged to get these door prizes. The arrangements are called permutations; the number of permutations of ten people, three at a time, is abbreviated \( 10P_3 \).

If the door prizes were all the same, though, it wouldn’t matter who got which one. All that matters is the number of trios of people who win. These collections are called combinations. The six permutations ABC, ACB, BAC, BCA, CAB, and CBA would count as one combination, because A, B, and C are the same prizes. The number of combinations of three people out of ten, written \( 10C_3 \), is only \( \frac{1}{6} \) of \( 10P_3 \), or 120.

Multiple-Stage Experiments
Tree diagrams can be useful for determining probabilities of more complicated experiments. For two flips of a coin, the possible outcomes and their probabilities can be shown with a tree diagram.
Chapter 10 • Probability (continued)

The expected value of an event is an average value found by multiplying the value of each possible event by its probability and adding the products. For example, the expected value of one spin on the spinner pictured would be found as shown:

$$\frac{1}{2}(-5) + \frac{1}{4}(2) + \frac{1}{4}(6) = -2.5 + 0.5 + 1.5 = -0.5$$

The expected value of the spinner is $-0.50.

Summary Problem

Imagine you have a bag with colored blocks, three blue and four red. What kinds of questions can be asked and answered involving choosing blocks from the bag?

Questions you can ask in your role as student to your student include:

- What is the probability of drawing a red block? A blue block?
- What is the probability of drawing two red blocks in a row? Do you need any more information?
- If the red blocks are worth $2 and the blue blocks are worth $5, what is the expected value of one draw?
- What values for each color block would give an expected value of $2 for one draw? Try to find several possibilities.

Sample Answers

The probability of drawing a red block is $\frac{3}{7}$; the probability of drawing a blue block is $\frac{4}{7}$. To find the probability of drawing two red blocks in a row, you must know whether the block will be replaced after is drawn. The probability of drawing two red blocks with replacement is $\frac{3}{7} \cdot \frac{3}{7} = \frac{9}{49}$, whereas the probability of drawing two red blocks without replacement is $\frac{3}{7} \cdot \frac{2}{6} = \frac{6}{42}$, or $\frac{3}{7}$. If red blocks are worth $2 and blue blocks are worth $5, the expected value of one draw is $\frac{3}{7}(2) + \frac{4}{7}(5) = \frac{6}{7} + \frac{20}{7} = \frac{26}{7} \approx 3.14$, or $3.14$. To have an expected value of $2 per draw, many combinations are possible. Some are $3.50 for red, $0 for blue; $2.75 for red, $1.00 for blue; $2.50 for red, and $8.00 for blue.
Chapter 10 • Review Exercises

Name ___________________________  Period ____________  Date ______________

1. (Lessons 10.1, 10.2) Sharon bought a bag of colored balloons for a party. The bag contained 9 white, 39 blue, 24 pink, 21 green, and 57 yellow balloons.
   a. Determine the percentage of each color of balloon, and use that information to make a circle graph and a relative frequency bar graph.
   b. What percentage of the balloons are neither pink nor white?
   c. If Sharon reaches in the bag and pulls out one balloon at random, what is the probability that it will be green?

2. (Lesson 10.3) Consider the figure at right.
   a. If you randomly plot a point in the large rectangle, what is the theoretical probability that it will land in the shaded region?
   b. Suppose you randomly plot many points, and 40 of them land in the shaded region. Estimate the total number of points plotted.

3. (Lesson 10.4) A high school is having a lottery, where three different digits are chosen from the digits 0–9 to create the winning number. To win, you must correctly guess the winning number.
   a. Suppose the winning guess must have the same three digits, in the same order, as the winning number. How many three-digit numbers can be made from the digits 0 to 9, where no digit is used twice? What is the probability of winning in this case?
   b. Now suppose the winning guess must have the same three digits as the winning number, but the digits can be in any order. What is the probability of winning in this case?

4. (Lessons 10.5, 10.6) Brigham has a bag that contains seven numbered chips. There are five chips labeled with the number 7, and two chips with the number 4. He reaches into the bag and pulls out a chip, sets that chip aside, and then pulls one more chip out of the bag.
   a. What is \( P(7_2 \mid 7_1) \)? What is \( P(4_2 \mid 4_1) \)?
   b. Create a tree diagram to calculate the probabilities of different outcomes of Brigham’s two-draw experiment.
   c. What is \( P(4_1 \text{ and } 7_2) \)?
   d. The numbers Brigham draws can add up to 8, 11, or 14. What is the probability that the sum will be an even number?
   e. Find the expected value of the sum.
1. a. There are 150 balloons total. Find the percentage of each color by dividing the number of that color by 150. For example, \( \frac{21}{150} = 0.14 \), so 14% of the balloons are green. Multiply the percentage by 360 to find the angle measure of each sector. For example, \( 0.14 \times 360 = 50.4^\circ \). Sample graphs are shown below.

b. The pink and white balloons together make up 6% + 16% = 22% of the total, so the percentage of balloons that are neither pink nor white is 100% - 22% = 78%.

c. 14% of the balloons are green, so the probability that she pulls out a green one is 14%, or 0.14.

2. a. The area of the shaded region is 21 squares, and the area of the whole rectangle is 8 \( \times \) 14 = 112 squares. Therefore, the probability of a randomly plotted point landing in the shaded region is \( \frac{21}{112} \) or 0.1875.

b. Solve the proportion \( \frac{21}{112} = \frac{40}{x} \) for \( x \).

\[ \frac{112}{21} = \frac{x}{40} \]
[Invert the proportion.]
\[ 112 \cdot 40 = x \cdot 21 \]
Multiply both sides by 40.
\[ x = 213.3 \]

Approximately 213 points were plotted.

3. a. There are 10 choices for the first digit, 9 choices for the second digit, and 8 choices for the third digit, so the total number of three-digit numbers formed from 0 to 9 without repetition is \( 10 \cdot 9 \cdot 8 = 720 \). You can also calculate \( 10P_3 \). The probability of guessing a winning number is \( \frac{1}{720} \approx 0.001 \).

b. For each three-digit number, there are \( 3 \cdot 2 \cdot 1 = 6 \) ways to arrange the digits. Because the order of the digits does not matter, divide the number of permutations you found in 3a by 6, to get \( \frac{720}{6} = 120 \). You can also calculate \( 10P_3 \). The probability of guessing a winning number is \( \frac{1}{120} \approx 0.008 \).

4. a. \( P(72 | 71) \) means “the probability that Brigham draws a 7 on his second draw, given that he drew a 7 on his first draw.” If Brigham drew a 7 on the first draw, then there would be four 7’s and two 4’s left in the bag, for a total of six chips. Therefore, \( P(72 | 71) = \frac{4}{6} = \frac{2}{3} \). If Brigham drew a 4 on the first draw, then there would be five 7’s and one 4 left in the bag for his second draw, so \( P(4 | 41) = \frac{1}{6} \).

b. The first branch of the tree diagram indicates the probabilities of the possible outcomes for Brigham’s first draw, and the second branch shows the probabilities of outcomes for his second draw.

```
<table>
<thead>
<tr>
<th>1st Draw</th>
<th>2nd Draw</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(71)</td>
<td>P(71 and 72) = \frac{10}{21}</td>
</tr>
<tr>
<td>P(41)</td>
<td>P(41 and 42) = \frac{5}{21}</td>
</tr>
<tr>
<td>P(72)</td>
<td>P(72 and 42) = \frac{5}{21}</td>
</tr>
<tr>
<td>P(42)</td>
<td>P(42 and 41) = \frac{5}{21}</td>
</tr>
</tbody>
</table>
```

c. Multiply the probabilities along the branches leading to the outcome 41 and 72. \( P(41 \text{ and } 72) = \frac{2}{7} \cdot \frac{5}{6} = \frac{10}{42} = \frac{5}{21} \). or \( \frac{5}{21} \).

d. Add the probabilities of outcomes that give even sums. \( P(\text{sum is even}) = P(71 \text{ and } 72) + P(41 \text{ and } 42) = \frac{10}{21} + \frac{5}{21} = \frac{15}{21} = \frac{5}{7} \).

e. For each outcome, multiply the sum of the numbers by the probability of the outcome, and then find the sum of the results. Expected value = \( P(71 \text{ and } 72) \cdot 14 + P(71 \text{ and } 42) \cdot 11 + P(41 \text{ and } 42) \cdot 11 + P(41 \text{ and } 41) \cdot 8 = \frac{10}{21} \cdot 14 + \frac{5}{21} \cdot 11 + \frac{5}{21} \cdot 11 + \frac{1}{21} \cdot 8 = \frac{86}{7} \) which is approximately 12.3.
CHAPTER 11

Introduction to Geometry

Content Summary
Chapter 11 is a preview of geometry. Even so, in some places it uses linear equations. Students also focus on radical expressions and operations with radicals.

Synthetic Geometry
The original study of geometry is now called synthetic to distinguish it from the analytic geometry of coordinate systems, which developed much later. In the area of synthetic geometry, the book focuses on the Pythagorean Theorem and on similar figures—figures that are stretches or shrinks of each other by the same factor in every direction.

Analytic Geometry
Analytic geometry is the geometry of coordinate graphs, which use algebraic equations to represent geometric figures. Students have been working with analytic geometry all through this course. They've already seen how parallel lines have the same slope, and how that slope appears in the equations of the lines. In this chapter they see how the slopes and equations of perpendicular lines relate. They also see how to find the coordinates of midpoints of line segments. In considering how the Pythagorean Theorem translates into coordinate geometry, students learn how to work with square roots.

Trigonometry
When a geometric figure is stretched or shrunk uniformly, all angles retain their measures and all side lengths are multiplied by the same amount; therefore, the ratio of one side length to another remains the same. This stretching and shrinking produces similar figures, which have equal corresponding angles and proportional corresponding side lengths. The study of relationships between sides and angles of similar right triangles is part of trigonometry. In particular, in similar right triangles, the ratio of, for example, the length of the side opposite a particular acute angle to the length of the hypotenuse is the same, no matter what the enlargement. Each acute angle of a right triangle has several of these ratios associated with it. They're called trigonometric ratios.

The book considers three such ratios: the sine, the cosine, and the tangent. Students use calculators to find values of these ratios for various angles and, in reverse, angles that correspond to various given ratios. They thus preview the deeper consideration of trigonometry found in Discovering Advanced Algebra.
Chapter 11 • Introduction to Geometry (continued)

Trigonometric Functions

For acute angle A in a right angle, the trigonometric functions are

- sine of angle $A = \frac{\text{length of opposite leg}}{\text{length of hypotenuse}}$ or $\sin A = \frac{o}{h}$
- cosine of angle $A = \frac{\text{length of adjacent leg}}{\text{length of hypotenuse}}$ or $\cos A = \frac{a}{h}$
- tangent of angle $A = \frac{\text{length of opposite leg}}{\text{length of adjacent leg}}$ or $\tan A = \frac{o}{a}$

![diagram of right triangle]

Summary Problem

You and your student might revisit this problem, adapted from Exercise 7 in Lesson 11.1, several times while working through the chapter.

What can you say about the quadrilateral with vertices ($-5, 0$), (1, 4), (6, 3), and ($-3, -3$)?

Questions you might ask in your role as student to your student include:
- Do any sides look parallel?
- Do any sides appear perpendicular?
- Can you confirm your conjectures?
- What are the equations of the four lines containing the sides of the quadrilateral?
- Do the diagonals meet at their midpoints?
- Can you find the lengths of the diagonals?
- Can you find the angle measures?

Sample Answers

Two of the sides are parallel ($y = -1 + \frac{3}{2}x$ and $3y = 10 + 2x$ both have slope $\frac{3}{2}$), and a third side is perpendicular to them ($2y = -15 - 3x$ with slope $-\frac{3}{2}$). The fourth side is contained in the line with equation $5y = 21 - x$. The diagonals lie on the lines with equations $y = 2(x - 1) + 4$ and $y = \frac{3}{11}(x + 5)$. Graphing and tracing indicates that the diagonals meet at about ($-0.6, 1.2$), which is not either of the midpoints, ($-1, 0.5$) or ($0.5, 1.5$). The diagonals have lengths $\sqrt{65}$ and $\sqrt{130}$. If you make a right triangle by dropping a perpendicular from the vertex at (1, 4) to the parallel side opposite, you can find that the sine of the angle at vertex (6, 3) is $\frac{\sqrt{13}}{\sqrt{26}} = \frac{1}{\sqrt{2}}$. Solve the equation $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = x$ to find that the angle is $45^\circ$. Therefore, the angle at (1, 4) will be $135^\circ$.
Chapter 11 • Review Exercises

1. (Lessons 11.1–11.3, 11.6) Plot the points $A(0, -2), B(-3, 1),$ and $C(3, 4)$.
   a. Find the equation of the line through $B$ that is parallel to $AC$.
   b. Find the midpoint of $AB$.
   c. Find the equation of the line containing the median of the triangle through $C$, and show that it is the perpendicular bisector of $AB$.
   d. Find the length of each side.
   e. What kind of triangle is $ABC$?
   f. Find the area of triangle $ABC$.

2. (Lesson 11.5) Rewrite each expression with as few square root symbols as possible, and no parentheses. Your final result should not have any perfect-square factors under a radical.
   a. $\left(2\sqrt{6}\right)\left(3\sqrt{2}\right)$
   b. $5\sqrt{3} - 2\sqrt{12}$
   c. $\frac{2\sqrt{18}}{3\sqrt{2}}$

3. (Lesson 11.7) The two triangles shown below are similar. Write a proportion and solve it to find $w$.

![Diagram of two similar triangles]

4. (Lessons 11.4, 11.8) Use the Pythagorean Theorem to find $x$. Then find the following ratios:

![Diagram of a right triangle]

a. $\tan A$  
   b. $\sin B$  
   c. $\cos B$
1. \[ \begin{array}{c}
\text{a. The segment } AC \text{ has slope 2, so write the equation of the line with slope 2 that passes through the point } (-3, 1). \text{ In point-slope form, this is } y - 1 = 2(x + 3), \text{ or } y = 2x + 7. \\
\text{b. The } x \text{-coordinate of the midpoint of } AB \text{ is }
\frac{0 + 5}{2} = 2.5, \text{ and the } y \text{-coordinate is }
\frac{-3 + 3}{2} = 0. \text{ The midpoint is } (2.5, 0). \\
\text{c. Find the equation of the line through the points } (3, 4) \text{ and } (-1.5, -0.5). \text{ The slope is } 
\frac{4 - (-0.5)}{3 - (-1.5)} = \frac{4.5}{4.5} = 1, \text{ so the equation is } y - 4 = 1(x - 3), \text{ or } y = x + 1. \text{ The slope of } AB \text{ is } -1 \text{ and the slope of the median is 1, so the product of the slopes is } -1. \text{ Therefore the two lines are perpendicular. The line } y = x + 1 \text{ is the perpendicular bisector of } AB \text{ because it is perpendicular to } AB \text{ and it passes through the midpoint of } AB. \\
\text{d. (Lesson 11.6) Use the distance formula } 
\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2} \text{ to calculate each length. For example: } 
AB = \sqrt{(-3 - 0)^2 + [1 - (-2)]^2} = \sqrt{9 + 9} = 3\sqrt{2}. \text{ And } AB = \sqrt{(-3 - (-3))^2 + (3 - 3)^2} = 0. \text{ For help in changing } \sqrt{18} \text{ to } 3\sqrt{2}, \text{ look ahead to Exercise 2. } \\
\text{Compute the other two lengths in a similar fashion. The lengths are } 
AC = \sqrt{45}, \text{ or } 3\sqrt{5}; \text{ and } BC = \sqrt{45}, \text{ or } 3\sqrt{5}. \\
\text{e. Triangle } ABC \text{ is a isosceles because } AC = BC. \\
\text{f. Draw a rectangle around triangle } ABC \text{ as shown on the graph for 1a. The rectangle has area 36. The right triangles with hypotenuses } BC \text{ and } AC \text{ have areas } 0.5(3)(6), \text{ or } 9 \text{ square units. The smaller right triangle has area } 0.5(3)(3), \text{ or } 4.5 \text{ square units. Subtract the areas of the triangles from the area of the rectangle: } 36 - (9 + 9 + 4.5) = 13.5. \text{ The area of triangle } ABC \text{ is } 13.5 \text{ square units.}
\end{array} \]